Boosting LASSO by Linear Embedding

Ashkan Panahi and Mats Viberg
Signal Processing Group, Chalmers University, Sweden
{*ashkanp, viberg*}@ *chalmers.se*

***Abstract*—Compressed Sensing (CS) has been successfully applied to the problem of single-snapshot Direction of Arrival (DOA) estimation. It is well known to be inconsistent when the true sources are close. The purpose of this work is to decrease the gap between CS applicable range and theoretical estimation bounds in high SNR regime. We show that the linear model of observation can be equivalently represented in a higher dimensional space, where there is a possibility to achieve better properties of the array manifold from CS point of view. We show the superior properties of this method by simulation.**

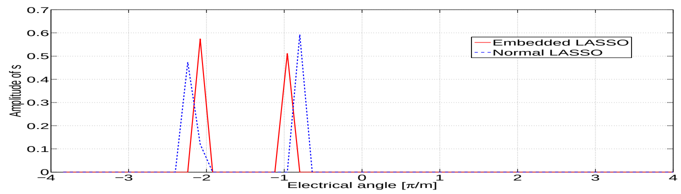

Fig. 1. Result of applying Noisy LASSO to the Embedded model with $\lambda = 1$. The parameters are given in the text.

I. Introduction and Mathematical Modeling

Let us assume the following noiseless model of Direction of Arrival (DOA) estimation [1] : $\mathbf{x} = \mathbf{A}\mathbf{s}$ where $\mathbf{x} \in \mathbb{C}^m$ is the observed vector and $\mathbf{s} \in \mathbb{C}^N$ is an unknown sparse vector. Moreover, the dictionary matrix $\mathbf{A} = [\mathbf{a}_1 \ \mathbf{a}_2 \dots \ \mathbf{a}_N]$ consists of a number of samples of the array manifold. An estimate of $\mathbf{s}$ can be found by solving the LASSO optimization

$$\min_{\mathbf{s}} \frac{1}{2}\|\mathbf{x} - \mathbf{A}\mathbf{s}\|_2^2 + \lambda\|\mathbf{s}\|_1 \tag{1}$$

where $\|\mathbf{s}\|_1$ denotes the sum of absolute entries of $\mathbf{s}$ and λ controls the sparsity level of the estimate. When there is no observation noise, the LASSO solution for $\lambda \to 0$ coincides with the true one, provided that the dictionary $\mathbf{A}$ satisfies the so called Restricted Isometry Property (RIP) condition[2]. However, $\mathbf{A}$ practically violates RIP and LASSO is generally inconsistent for close sources [3]. In this case, a desired solution order can be obtained by adjusting λ, which normally leads to highly biased estimates.

To cope with inconsistency, note that the linear model above defines a linear subspace of the set of all pairs $(\mathbf{s}, \mathbf{x})$. This subspace can be embedded in a higher dimensional space. In particular, consider the models of the type

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{A}' & \mathbf{p} \\ \mathbf{q} & \mathbf{Q} \end{pmatrix} \begin{pmatrix} \mathbf{s}' \\ \mathbf{d} \end{pmatrix}, \tag{2a}$$

$$\mathbf{s}' = \mathbf{R}\mathbf{s}. \tag{2b}$$

If (2a) and (2b) are equivalent to the original model, then the estimation can be performed by solving (2a) and then using (2b) to find the solution of the original problem. If an equivalent model in (2a) further satisfy RIP, the above estimation procedure is also consistent.

As verifying the RIP condition is time consuming, we exploit the more restrictive coherence condition [2]. For simplicity, we consider a single extra dimension $\mathbf{d} \in \mathbb{C}$, so that $\mathbf{p} \in \mathbb{C}^{m\times 1}$ and $\mathbf{q} \in \mathbb{C}^{1\times m}$. Without loss of generality we take $\mathbf{Q} = 1$. Then, neglecting mathematical details, we conclude that $\mathbf{A} = (\mathbf{A}' - \mathbf{p}\mathbf{Q}^{-1}\mathbf{q})\mathbf{R}$ with a diagonal $\mathbf{R}$ for an equivalent model and the coherence μ can be obtained by

$$\mu = \max_{k\neq l} \frac{\left|(\mathbf{a}_k + q_k\mathbf{p})^H(\mathbf{a}_l + q_l\mathbf{p}) + q_k^* q_l\right|}{\sqrt{\|\mathbf{a}_k + q_k\mathbf{p}\|_2^2 + |q|_k^2}\sqrt{\|\mathbf{a}_k + q_l\mathbf{p}\|_2^2 + |q|_l^2}} \tag{3}$$

for an arbitrary choice of $\mathbf{p}$ and $\mathbf{q} = [q_1, q_2, \dots, q_N]$. Accordingly, we propose to minimize μ in (3) to select the best equivalent embedding. Finally the diagonal elements r_l of $\mathbf{R}$ can be found by $r_l = \|\mathbf{a}_l + q_l\mathbf{p}\|_2$, where more details are postponed to the final version. Once the embedding in (2b) is obtained from the above, the estimate of $\mathbf{s}'$ is found by solving LASSO with a proper choice of λ. As $\mathbf{s}$ and $\mathbf{s}'$ has the same support, the DOA estimates are directly found from the support of $\mathbf{s}'$.

II. Results and Conclusion

Figure 1 shows the amplitudes of the estimates for an adjusted value of λ in an extremely difficult case of finding directions of two sources by a half-wavelength uniform linear array with $m = 4$ sensors and $N = 100$ grid points. The sources are located at electrical angles $[-2\pi/m \quad -\pi/m]$, separated with completely destructive waveform values $[1 \ -1]$ respectively. The figure shows that in this worst case the embedded LASSO provides less bias. In another experiment we compute the average bias by zero-mean unit-variance Gaussian sources fixing all parameters to their previously mentioned value. A Monte Carlo simulation reveals that the average bias decreases from 0.0900 to 0.007 in terms of electrical angle. However, it is expected that a higher dimensional embedding provides a completely consistent estimate.

References

[1] D. Malioutov, "A sparse signal reconstruction perspective for source localization with sensor arrays," Master's thesis, MIT, July 2003.
[2] R. G. Baraniuk, "Compressive sensing [lecture notes]," *IEEE Signal Processing Mag.*, vol. 24, pp. 118–121, July 2007.
[3] A. Panahi and M. Viberg, "On the resolution of the lasso-based doa estimation method," in *Smart Antennas (WSA), 2011 International ITG Workshop on*, feb. 2011, pp. 1 –5.