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Dictionary Adaptation in Sparse Recovery Based on Different Types of Coherence

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Outline

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Introduction

Sparse Recovery in Signal Acquisition

Measurement of a Signal

Signal $\boldsymbol{x} \in \mathbb{R}^N$ is acquired by measurement matrix $\boldsymbol{\Phi} \in \mathbb{R}^{M \times N}$:

$$\boldsymbol{y} = \boldsymbol{\Phi}\boldsymbol{x} \quad \text{with} \quad M < N.$$

Sparse Representation

Signals can be sparsely represented with a dictionary $\boldsymbol{\Psi} \in \mathbb{R}^{N \times L}$:

$$\boldsymbol{x} = \boldsymbol{\Psi}\boldsymbol{\alpha} \quad \text{with} \quad N \leq L.$$

Under-determined System of Linear Equations with Sparse Solution

Sensing matrix is obtained by $\boldsymbol{\Phi}\boldsymbol{\Psi} = \boldsymbol{A} \in \mathbb{R}^{M \times L}$

$$\boldsymbol{y} = \boldsymbol{\Phi}\boldsymbol{\Psi}\boldsymbol{\alpha} = \boldsymbol{A}\boldsymbol{\alpha}$$

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Coherence-Based Optimization Criteria

Column Coherence of $\boldsymbol{A}$

$$\mu(\boldsymbol{A}) = \max_{i \neq j} \frac{|\langle \boldsymbol{a}_i, \boldsymbol{a}_j \rangle|}{\|\boldsymbol{a}_i\|_2 \|\boldsymbol{a}_j\|_2},$$

where $\boldsymbol{a}_i$ is the i-th column of $\boldsymbol{A}$.

- Often used where $\boldsymbol{x}$ is sparse itself ($\boldsymbol{\Psi} = \mathbf{I}$)
- Several approaches optimize this property e.g. [Elad 2007]

Row Coherence of $\boldsymbol{\Phi}$ with respect to Columns of $\boldsymbol{\Psi}$

$$\mu(\boldsymbol{\Phi}, \boldsymbol{\Psi}) = \max_{i,j} \frac{|\langle \boldsymbol{\phi}_i, \boldsymbol{\psi}_j \rangle|}{\|\boldsymbol{\phi}_i\|_2 \|\boldsymbol{\psi}_j\|_2},$$

where $\boldsymbol{\phi}_i$ is the i-th row of $\boldsymbol{\Phi}$ and $\boldsymbol{\psi}_j$ is the j-th column of $\boldsymbol{\Psi}$.

- Motivated by measurement process

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Best Antipodal Spherical Codes

Denotation

$\Omega_{\mathcal{N}}(\mathbf{0}, 1)$	Unit sphere centered at the origin $\mathbf{0}$ of $\mathbb{R}^{\mathcal{N}}$.
$C_s(\mathcal{N}, \mathcal{M})$ $= \{\boldsymbol{s}_m\}_{m=1}^{\mathcal{M}}$	A spherical code is a set of $\mathcal{M}$ points $\boldsymbol{s}_m$ placed on $\Omega_{\mathcal{N}}(\mathbf{0}, 1)$.
$C_{\mathrm{bs}}(\mathcal{N}, \mathcal{M})$	Best spherical codes (BSC) maximize the minimal Euclidean (or angular) distance $d_{ml} = \lvert\boldsymbol{s}_m - \boldsymbol{s}_l\rvert$ and *minimize the maximal inner product* of code words.
$C_{\mathrm{bas}}(\mathcal{N}, \mathcal{M})$	For best antipodal spherical codes (BASC), the antipodal of each code word is also a code word: $\boldsymbol{s}_m \in C_{\mathrm{bas}}(\mathcal{N}, \mathcal{M}) \iff -\boldsymbol{s}_m \in C_{\mathrm{bas}}(\mathcal{N}, \mathcal{M})$ Thus, *maximal coherence is minimized*.
$\underline{\boldsymbol{u}} = \frac{\boldsymbol{u}}{\lvert\boldsymbol{u}\rvert}$	Underlined denotation of normalized vectors.

Best Antipodal Spherical Codes

General Ideas

Generating BSC

- Points of a $C_s(\mathcal{N}, \mathcal{M})$ can be considered as charged particles acting in some field of repelling forces.
- Particles will move until total potential of system approaches some local minimum.

Optimize Coherence with BASC

- Consider also the antipodals of $C_s(\mathcal{N}, \mathcal{M})$.
- Obtain $C_{\text{bas}}(\mathcal{N}, \mathcal{M})$ by generating BSC (updating antipodals).

Optimize $\mu(\Phi, \Psi)$ with BASC

- Points corresponding to Ψ and their antipodals are fixed.
- Particles of Φ are free to be moved (updating antipodals).

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Best Antipodal Spherical Codes

Generating BSC

Generalized Potential Function

$$g(C_s(\mathcal{N},\mathcal{M})) = \sum_{m<l} |\boldsymbol{s}_m - \boldsymbol{s}_l|^{-(\nu-2)} \quad \text{with} \quad \nu \in \mathbb{N}\ (\nu > 2),$$

attains a global minimum by a BSC if $\nu \to \infty$.

Lagrangian multipliers

A global minimum can be expressed by an equilibrium:

$$\left\{ \underline{\boldsymbol{s}}_m = \sum_{l \neq m} \frac{\underline{\boldsymbol{s}}_m - \underline{\boldsymbol{s}}_l}{|\underline{\boldsymbol{s}}_m - \underline{\boldsymbol{s}}_l|^{\nu}} \right\}_{m=1}^{\mathcal{M}}$$

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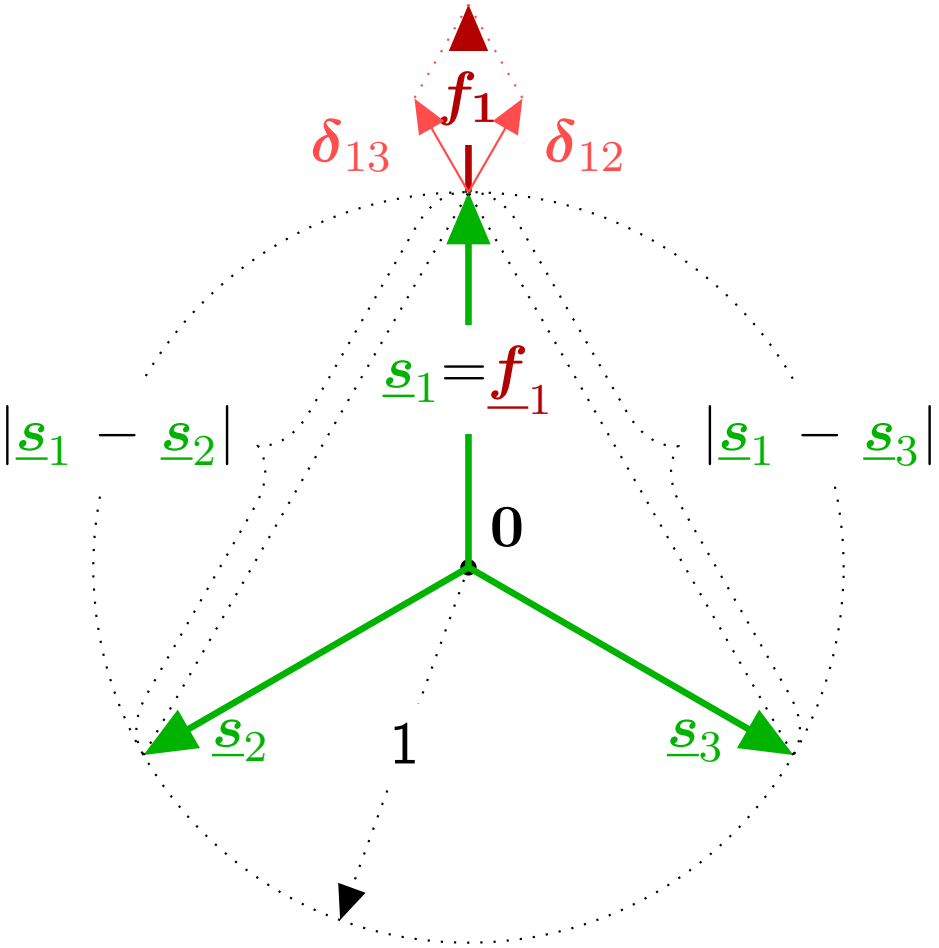
Best Antipodal Spherical Codes

Generating BSC - Illustration



$$C_{bs}(2,3) = \{\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3\}.$$

Best Antipodal Spherical Codes

Generating BSC - The Iterative Process

Mapping

$$\boldsymbol{P}[C_s(\mathcal{N},\mathcal{M})] = \left\{\underline{\boldsymbol{s}}_m + \alpha\underline{\boldsymbol{f}}_m\right\}_{m=1}^{\mathcal{M}} \quad \text{with} \quad \alpha \in \mathbb{R}$$

Iterative Process for Coherence Optimization

$$C_s(\mathcal{N},\mathcal{M})^{(k+1)} = \boldsymbol{P}(C_s(\mathcal{N},\mathcal{M})^{(k)}); \; k = 0, 1, \ldots$$

converges for a small enough "damping factor" α.

Optimization Strategy

For increasing ν, the iterative process is continuously applied.

Best Antipodal Spherical Codes

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Best Antipodal Spherical Codes

Generating BSC - Algorithm

procedure BSC-BASED MIN-DISTANCE OPTIMIZATION($\mathcal{N}, \mathcal{M}$)
  $C_s \leftarrow$ spherical seed ▷ Random spherical code
  while $\nu < \nu_{max}$ **do**
    while $i < i_{max}$ **AND** FixedPointFound $=$ false **do**
      for $m = 1$ to $\mathcal{M}$ **do**
        $\boldsymbol{f}_m \leftarrow \underline{\boldsymbol{s}}_m = \underline{\sum_{l \neq m} \frac{\underline{\boldsymbol{s}}_m - \underline{\boldsymbol{s}}_l}{|\underline{\boldsymbol{s}}_m - \underline{\boldsymbol{s}}_l|^{\nu}}}$ ▷ Calculate generalized force
      end for
      $\{\boldsymbol{s}_m\}_{m=1}^{\mathcal{M}} \leftarrow \left\{\underline{\underline{\boldsymbol{s}}_m + \alpha \underline{\boldsymbol{f}}_m}\right\}_{m=1}^{\mathcal{M}}$ ▷ Apply force
    end while
  end while
  return $\{\underline{\boldsymbol{s}}_m\}_{m=1}^{\mathcal{M}}$
end procedure

Coherence Optimization

Generating BASC - Algorithm

procedure BASC-BASED COHERENCE OPTIMIZATION($\mathcal{N}, \mathcal{M}$)

$C_s \leftarrow$ spherical seed ▷ Random spherical code

$C_{as} \leftarrow [C_s \quad -C_s]$ ▷ Antipodal spherical code

while $\nu < \nu_{max}$ **do**

while $i < i_{max}$ **AND** FixedPointFound $=$ false **do**

for $m = 1$ to $\mathcal{M}$ **do**

$$\boldsymbol{f}_m \leftarrow \underline{\boldsymbol{s}}_m = \sum_{\substack{l \neq m \\ l \neq m+\mathcal{M}}} \frac{\underline{\boldsymbol{s}}_m - \underline{\boldsymbol{s}}_l}{|\underline{\boldsymbol{s}}_m - \underline{\boldsymbol{s}}_l|^\nu}$$ ▷ Calculate generalized force

end for

$$\{\boldsymbol{s}_m\}_{m=1}^{\mathcal{M}} \leftarrow \left\{\underline{\boldsymbol{s}}_m + \alpha \underline{\boldsymbol{f}}_m\right\}_{m=1}^{\mathcal{M}}$$ ▷ Apply force

$$\{\boldsymbol{s}_m\}_{m=1+\mathcal{M}}^{2\mathcal{M}} \leftarrow \{-\underline{\boldsymbol{s}}_m\}_{m=1}^{\frac{M}{2}}$$ ▷ Update antipodals

end while

end while

return $\{\underline{\boldsymbol{s}}_m\}_{m=1}^{\mathcal{M}}$ ▷ Return non-antipodals

end procedure

Coherence Optimization

Optimizing $\mu(\mathbf{\Phi}, \mathbf{\Psi})$- Algorithm

procedure BASC-BASED $\mu(\mathbf{\Phi}, \mathbf{\Psi})$ OPTIMIZATION$(\mathbf{\Psi}, \mathcal{N}, \mathcal{M})$

$C_s \leftarrow$ spherical seed ▷ Random spherical code

$C_{as} \leftarrow [C_s \quad -C_s \quad \mathbf{\Psi} \quad -\mathbf{\Psi}]$ ▷ Antipodal spherical code

while $\nu < \nu_{max}$ **do**

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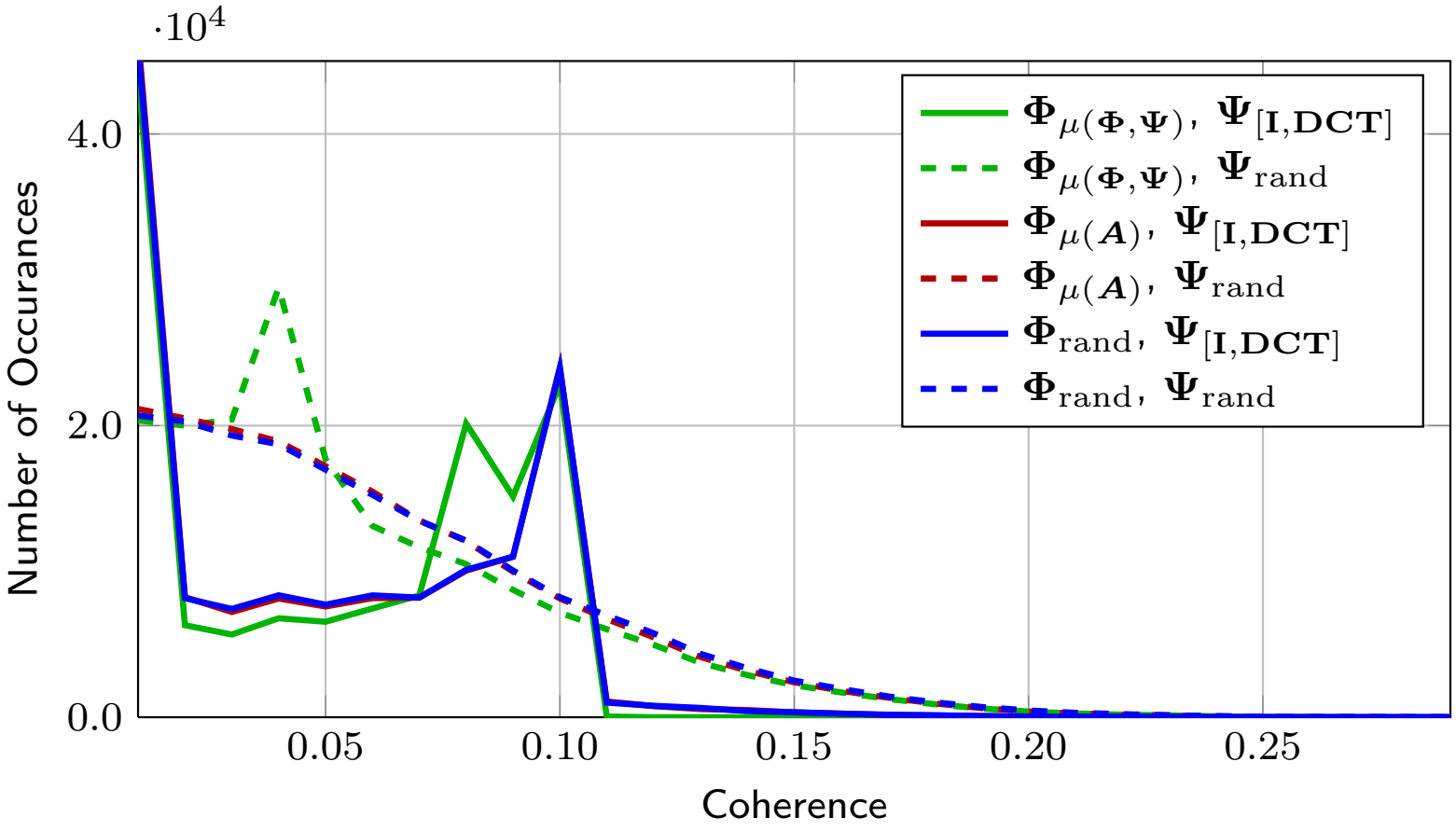
Outline

Numerical Evaluation

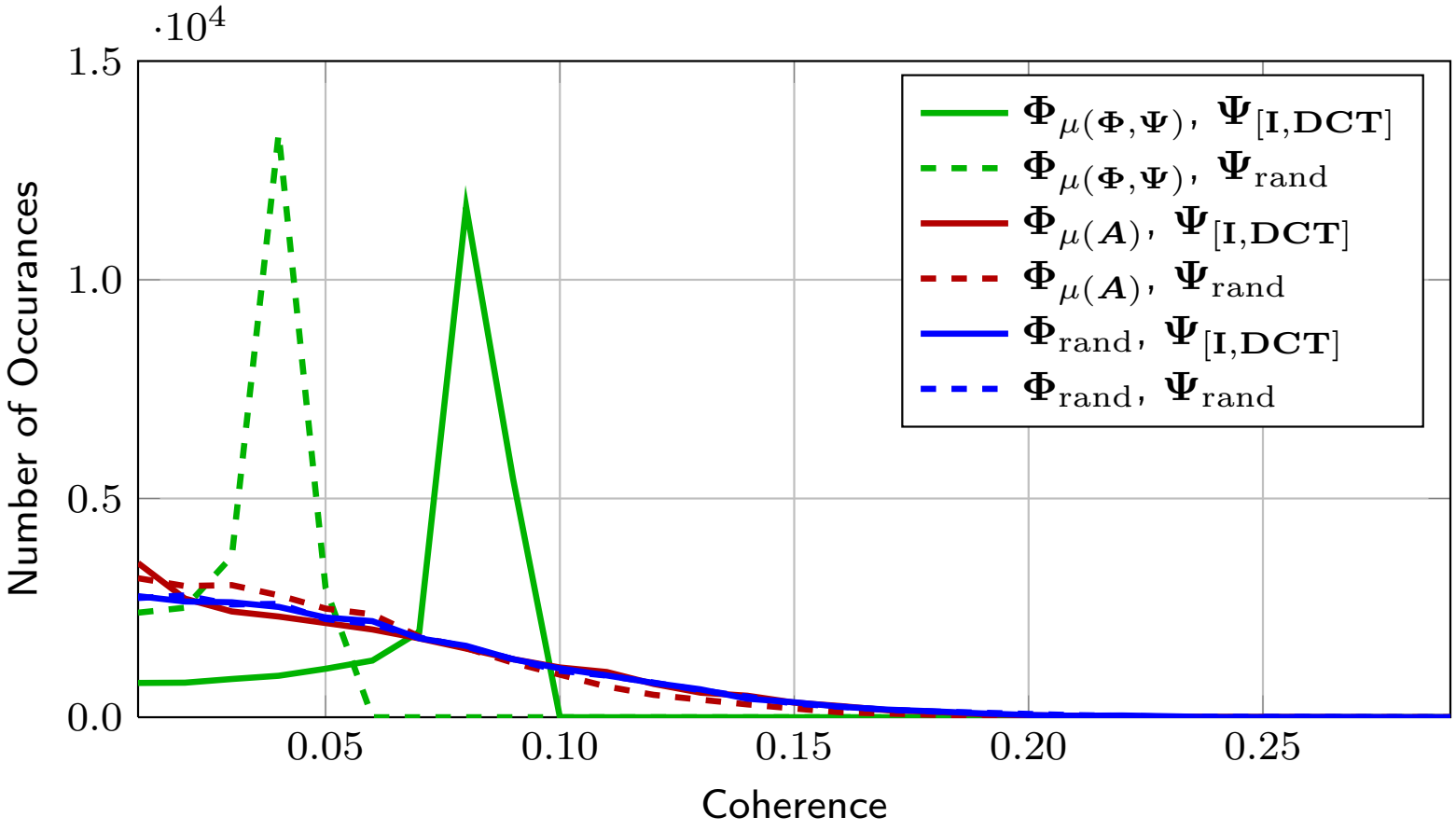
Verify Success of Optimizations - $\mu(\mathbf{\Phi}, \mathbf{\Psi})$



Coherence distribution of $\left[\mathbf{\Phi}^T, \mathbf{\Psi}\right]$ for $M = 30$, $N = 200$ and $L = 400$.

Numerical Evaluation

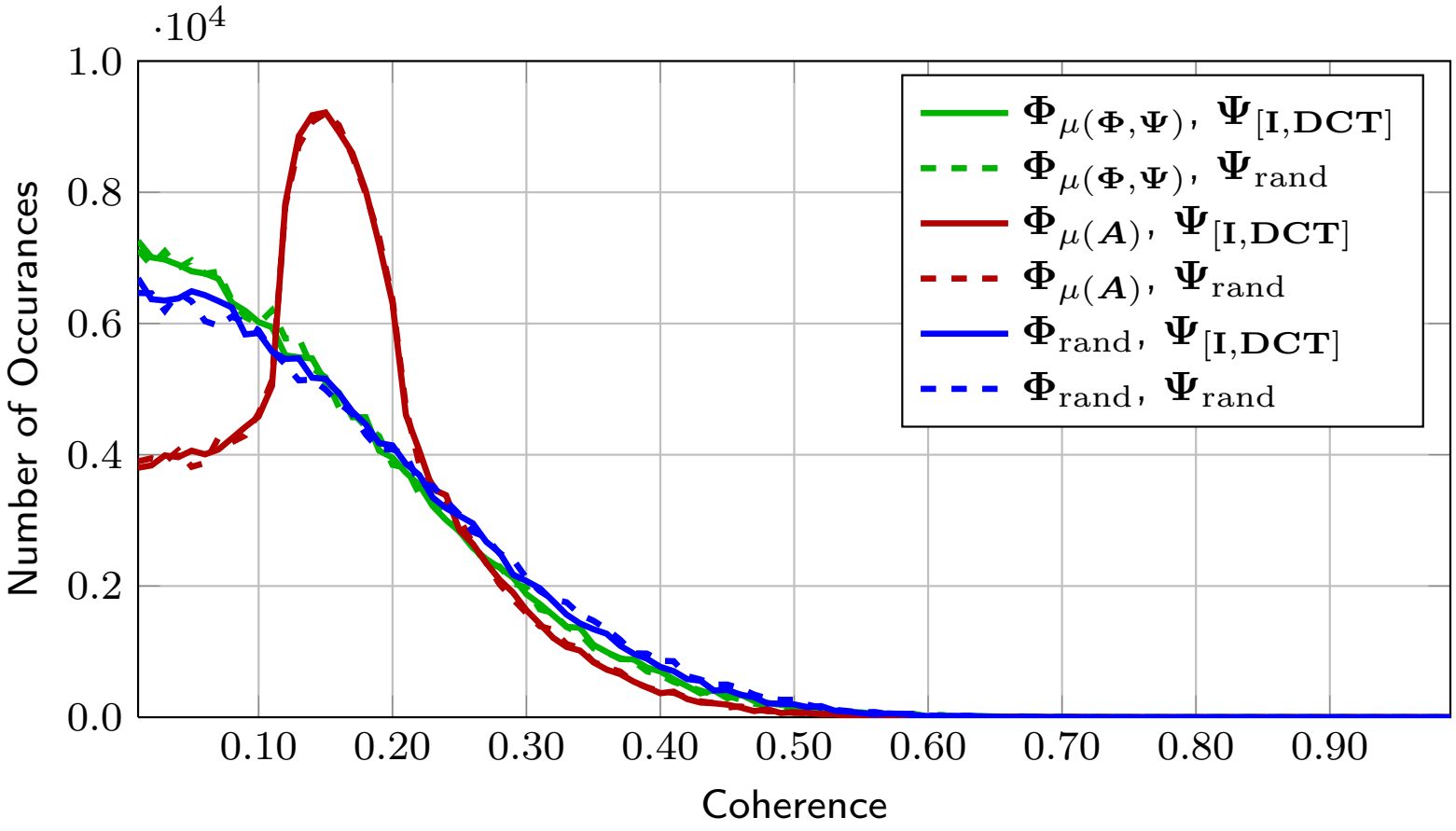
Verify Success of Optimizations - $\mu(\mathbf{\Phi}, \mathbf{\Psi})$ - 2



Coherence distribution of $\left[\mathbf{\Phi}^T, \mathbf{\Psi}\right]$ for $M = 30$, $N = 200$ and $L = 400$.
Intra column coherence of $\mathbf{\Psi}$ is removed.

Numerical Evaluation

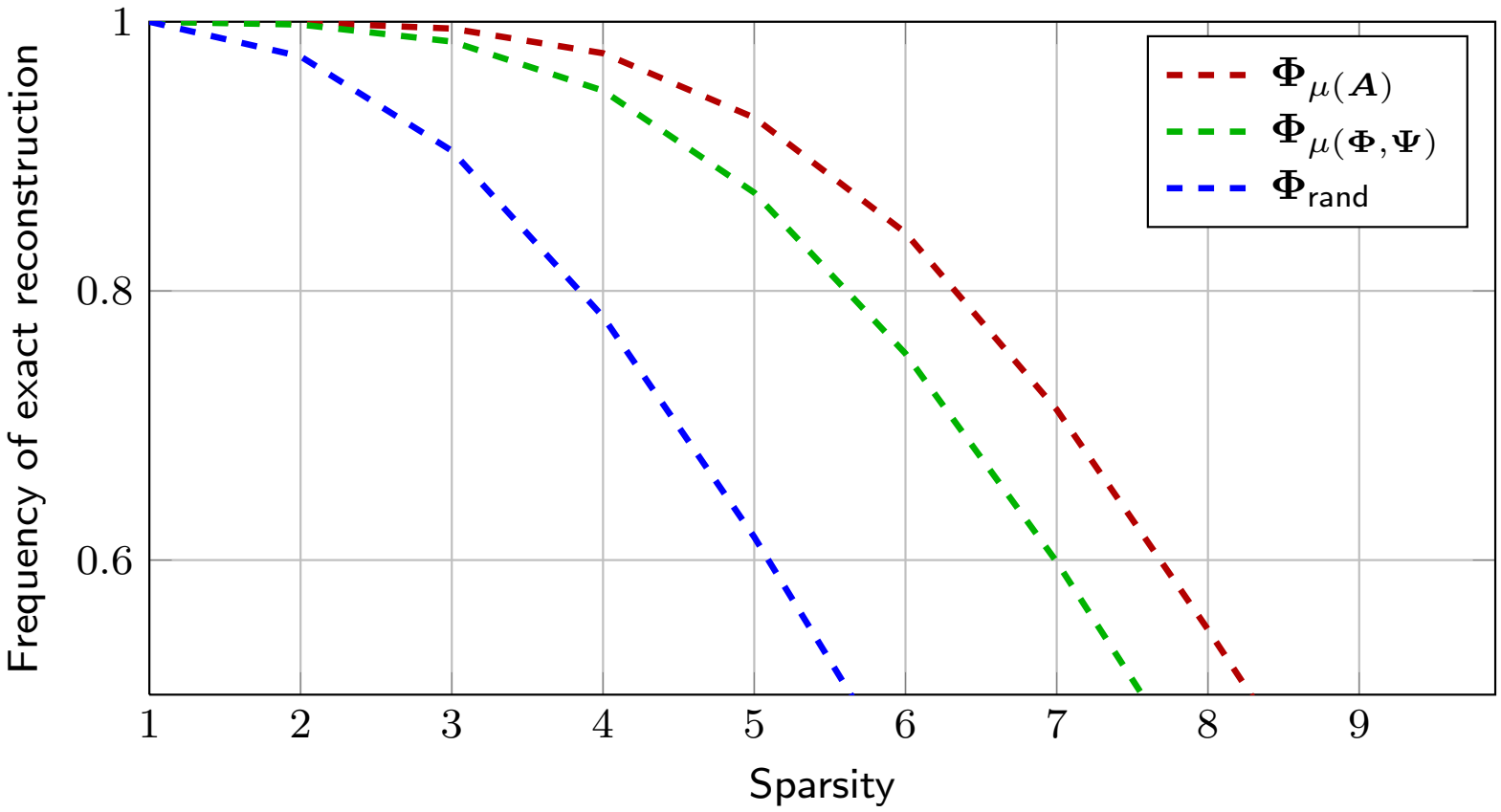
Verify Success of Optimizations - $\mu(\boldsymbol{A})$



Coherence distribution of $\boldsymbol{A}$ for $M = 30$, $N = 200$ and $L = 400$.

Numerical Evaluation

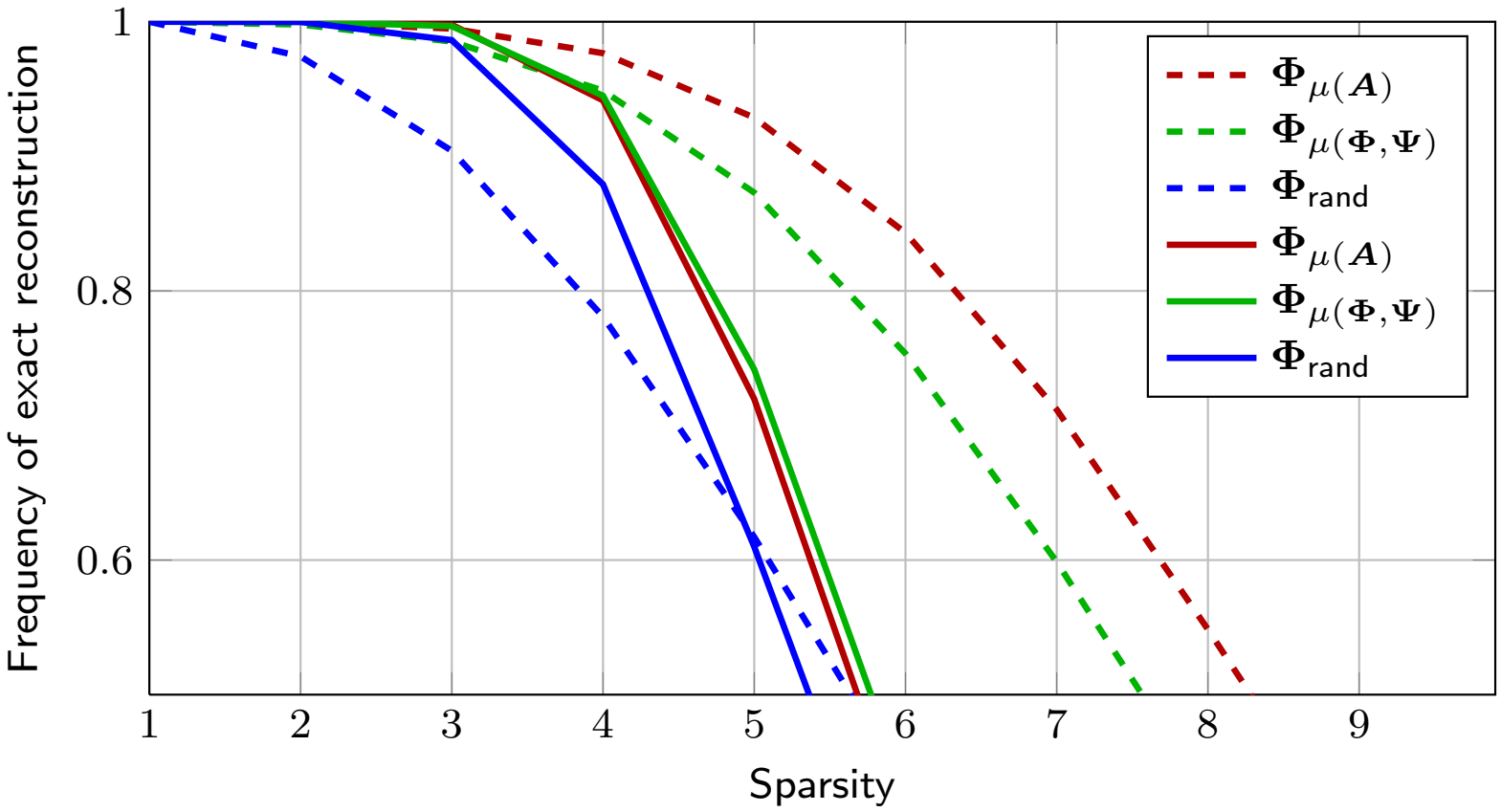
Evaluation of Effectiveness



Frequency of exact reconstruction with $\mathbf{\Psi}_{[\mathbf{I},\mathbf{DCT}]}$ for $M = 30$, $N = 200$ and $L = 400$. OMP is used for dashed lines.

Numerical Evaluation

Evaluation of Effectiveness



Frequency of exact reconstruction with $\mathbf{\Psi}_{[\mathbf{I},\mathbf{DCT}]}$ for $M = 30$, $N = 200$ and $L = 400$. OMP is used for dashed lines, BP for solid lines.

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Conclusion

Contribution

- Proposed a method to optimize $\mu(\mathbf{\Phi}, \mathbf{\Psi})$ using BASC
- Compared the effectiveness of optimization strategies

Numerical Evaluation

- Both optimization criteria lead to improved results.
- Algorithms relying on $\mu(\boldsymbol{A})$ favor the optimization of $\mu(\boldsymbol{A})$.
- Optimization of $\mu(\mathbf{\Phi}, \mathbf{\Psi})$ may be considered elsewise.

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End of Presentation

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THE END

Do you have any additional questions?