Tensor-Based Dictionary Learning for Multidimensional Sparse Recovery: the K-HOSVD

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In many applications of compressive sensing, the dictionary providing the sparse description is partially or entirely unknown. It has been shown that dictionary learning algorithms are able to estimate the basis vectors from a set of training samples. In some applications the dictionary is multidimensional, e.g., when estimating jointly azimuth and elevation in a 2-D direction of arrival (DOA) estimation context. In this paper we show that existing dictionary learning algorithms can be extended to exploit this structure, thereby providing a more accurate estimate of the dictionary. As an example we choose the well-known K-SVD algorithm [1], propose a tensor extension and show its improved performance numerically.

Consider a sparse recovery problem given by

$$\boldsymbol{Y} = \boldsymbol{A} \cdot \boldsymbol{S} + \boldsymbol{N}, \tag{1}$$

where $\boldsymbol{Y} \in \mathbb{C}^{M \times T}$ denotes a set of T subsequent observations from M channels, $\boldsymbol{A} \in \mathbb{C}^{M \times N}$ is the dictionary containing N atoms, $\boldsymbol{S} \in \mathbb{C}^{N \times T}$ is a sparse matrix which contains at most K non-zero elements per column, and $\boldsymbol{N} \in \mathbb{C}^{M \times N}$ models the additive measurement noise.

In some applications the dictionary obeys a multidimensional structure, which allows us to write it as

$$\boldsymbol{A} = \boldsymbol{A}_1 \otimes \boldsymbol{A}_2 \tag{2}$$

where $\boldsymbol{A}_k \in \mathbb{C}^{M_k \times N_k}$ denotes the dictionaries for the separate dimensions $k = 1, 2$ and $\otimes$ is the Kronecker product. An example for (2) is 2-D direction of arrival (DOA) estimation using a 2-D separable antenna array such as a uniform rectangular array (URA) [2].

Depending on the application, the actual dictionary may be entirely unknown (as it is common in image processing) or partially unknown (as in DOA estimation where the true source locations are not exactly on the grid points we chose for discretizing the array manifold).

Existing dictionary learning algorithms estimate $\boldsymbol{A}$ and $\boldsymbol{S}$ jointly from $\boldsymbol{Y}$ by exploiting the fact that $\boldsymbol{S}$ is sparse. A well-known example is given by the K-SVD algorithm [1]. It alternates between two steps. Firstly, a sparse recovery algorithm is used to recover the support set in $\boldsymbol{S}$. Given the active support set, all active atoms are then updated sequentially. To update one atom, the contribution from all other atoms is subtracted. Since the remaining matrix is ideally rank-one, the best approximation for the current atom is found by a rank-one truncated SVD.

This idea can readily be extended to multiple dimensions. After subtracting all active atoms but one, the remaining matrix is given by

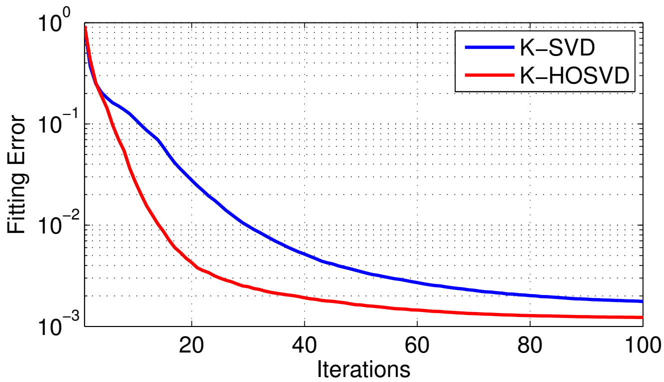

Fig. 1. Convergence behavior of K-SVD against K-HOSVD

$$\boldsymbol{Y} \approx (\boldsymbol{a}_1(i_1) \otimes \boldsymbol{a}_2(i_2)) \cdot \boldsymbol{s}(i)^{\mathrm{T}}, \tag{3}$$

where $\boldsymbol{a}_n(i_n)$ is the i_n-th column of $\boldsymbol{A}_n$, $\boldsymbol{s}(i)^{\mathrm{T}}$ is the i-th row of $\boldsymbol{S}$, and $i = (i_1 - 1) \cdot M_2 + i_2$. Reshaping the $M \times T$ matrix $\boldsymbol{Y}$ into an $M_1 \times M_2 \times T$ tensor $\boldsymbol{\mathcal{Y}}$, (3) becomes

$$\boldsymbol{\mathcal{Y}} \approx \boldsymbol{a}_1(i_1) \circ \boldsymbol{a}_2(i_2) \circ \boldsymbol{s}(i), \tag{4}$$

where $\circ$ denotes the outer (tensor) product. Equation (4) shows that we have reformulated the original rank-one matrix approximation problem into a rank-one tensor approximation problem. Finding the optimal rank-one approximation of a tensor in general requires iterative algorithms. However, we apply the low-complexity closed-form solution given by the truncated Higher-Order SVD (HOSVD), since it already provides a very good estimate for moderate to high SNRs.

In Figure 1 we present a numerical example where we trained a random dictionary for $M_1 = M_2 = 4$ and $N_1 = N_2 = 6$ and a sparsity of $K = 3$ based on $T = 100$ training samples at an SNR of 30 dB. We show fitting error defined as $\left\| \boldsymbol{Y} - \hat{\boldsymbol{A}} \cdot \hat{\boldsymbol{X}} \right\|_{\mathrm{F}}^2 / \left\| \boldsymbol{Y} \right\|_{\mathrm{F}}^2$ where $\left\| \cdot \right\|_{\mathrm{F}}$ denotes the Frobenius norm. We observe that the K-HOSVD converges faster and yields a better fit. This is due to the fact that the structure of the dictionary is explicitly exploited.

References

[1] M. Aharon, M. Elad, and A. Bruckstein, "The K-SVD: An algorithm for designing of overcomplete dictionaries for sparse representation," *IEEE Trans. on Signal Processing*, vol. 54, no. 11, pp. 4311–4322, 2006.

[2] D. Nion and N. D. Sidiropoulos, "Tensor Algebra and Multidimensional Harmonic Retrieval in Signal Processing for MIMO Radar", *IEEE Trans. on Signal Processing*, vol. 58, no. 11, pp. 5693–4705, 2010.