

Two approaches to remote sensing via ℓ_1 -minimization

Max Hügel

University of Bonn and RWTH Aachen

Holger Rauhut

RWTH Aachen

Thomas Strohmer

UC Davis

CoSeRa 2013

September 17, 2013

① General setup

② Random array

③ Deterministic array

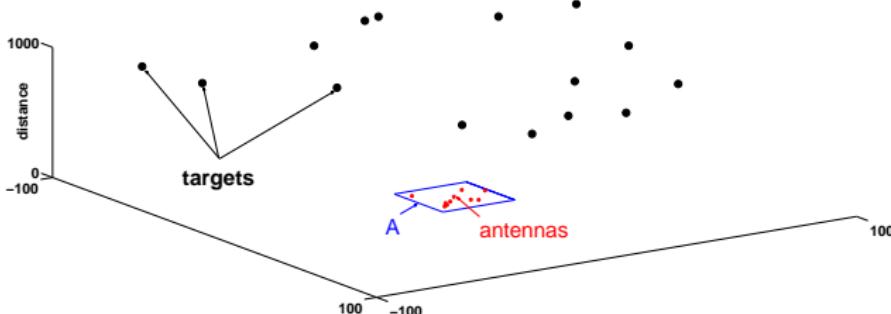
Basic setup

Sensing device:

- Square array $[-A/2, A/2]^2$ with aperture $A > 0$,
- antenna elements $a_1, \dots, a_n \in [-A/2, A/2]^2$.

Target image: Point targets $x := \sum_{\ell=1}^s x_\ell \delta_{p_\ell}$ at distance $z_0 > 0$,

- $\{x_\ell\}_{\ell=1,\dots,s} \subset \mathbb{C}$ reflectivities,
- $\{p_\ell\}_{\ell=1,\dots,s} \subset [-L, L]^2 \times \{z_0\}$ positions, $L > 0$ target domain.



Transmit waveform

- **Transmit waveform:** Monochromatic wave of wavelength $\lambda > 0$;
- $a_k \in [-A/2, A/2]^2$ transmits, spatial part recorded in $p_\ell \in [-L, L]^2 \times \{z_0\}$ given by

$$G(a_k, p_\ell) := \frac{\exp\left(\frac{2\pi i}{\lambda} \|a_k - p_\ell\|_2\right)}{4\pi \|a_k - p_\ell\|_2}.$$

- Under **far field assumption** ($z_0 \gg A + L$) and with $p_\ell = (\xi_\ell, \eta_\ell, z_0)$:

$$G(a_k, p_\ell) \approx \tilde{G}(a_k, p_\ell) := \frac{\exp\left(\frac{\pi i}{\lambda z_0} \|a_k - (\xi_\ell, \eta_\ell)\|_2^2\right)}{4\pi z_0}$$

Measurement model

- **TR-mode:** one antenna transmits, whole array records echo.
- $a_k \in [-A/2, A/2]^2$ transmits, $a_m \in [-A/2, A/2]^2$ receives,
echo from scene $x = \sum_{\ell=1}^s x_\ell \delta_{p_\ell}$

$$y_{km} := \sum_{\ell=1}^s \tilde{G}(a_k, p_\ell) \tilde{G}(p_\ell, a_m) x_\ell. \quad (\textbf{Born approximation})$$

- Measure **Fourier coefficients** of x at spatial frequencies

$$f_{km} := \frac{1}{\lambda z_0} (a_k + a_m).$$

Random antenna locations

- Choose n antenna positions $a_1, \dots, a_n \in [-A/2, A/2]^2$ independently and uniformly at random.
- Discretize target space $[-L, L]^2 \times \{z_0\}$ into grid $(p_j)_{j=1,\dots,N}$ of meshsize

$$d_0 = \lambda z_0 / A \quad (\textbf{Rayleigh resolution}).$$

- Scene represented by vector $(x_j)_{j=1,\dots,N}$ of reflectivities.
Assumption: x **s-sparse** with $s \ll N$.

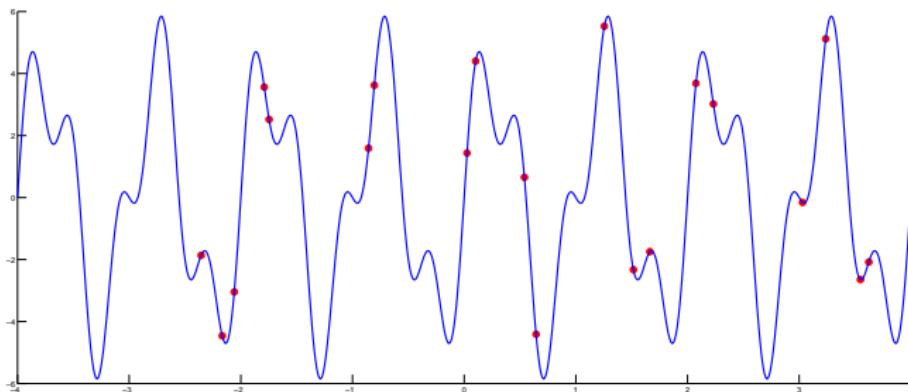
Spectrum interpolation via ℓ_1 -minimization

- **Recall:** Measure Fourier coefficients $y = Ax \in \mathbb{C}^{n^2}$ at random frequencies $1/(\lambda z_0)(a_k + a_m)$.
- **I.e.:** Undersample full spectrum of x at random locations.
- **Goal:** Recover x , i.e. **interpolate** the full spectrum from knowledge at random frequencies.
- **Algorithm:** Suppose measure $y = Ax + z$ with complex Gaussian noise $z \sim \mathcal{N}_{\mathbb{C}}(0, \sigma^2)$. Solve **noise-constrained** ℓ_1 -minimization:

$$\arg \min_{v \in \mathbb{C}^N} \|v\|_1 \quad \text{subject to } \|Av - y\|_2 \leq \text{const } \sigma n.$$

Random spectrum sampling

Compressed Sensing \cong Spectrum interpolation



A rigorous CS result

Theorem (H., Rauhut, Strohmer 2013)

$x \in \mathbb{C}^N$ fixed scene, measure $y = Ax + z \in \mathbb{C}^{n^2}$ where
 $z \sim \mathcal{N}_{\mathbb{C}}(0, \sigma^2)$. Suppose that for $\epsilon > 0$

$$n^2 \geq Cs \log^2 \left(\frac{N}{\epsilon} \right),$$

$C > 0$ universal constant. If discretization meshsize satisfies
 $d_0 = \lambda z_0 / A$, then with probability at least $(1 - \epsilon)^2$, solution \hat{x} to

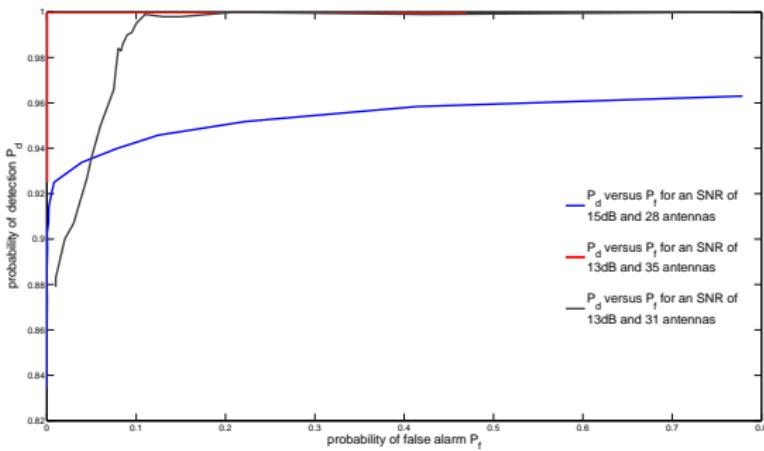
$$\arg \min_{v \in \mathbb{C}^N} \|v\|_1 \quad \text{subject to } \|\tilde{A}v - y\|_2 \leq \sigma n \log(1/\epsilon)$$

satisfies

$$\|\hat{x} - x\|_2 \leq C_1(\sqrt{s}\sigma + \sigma_s(x)_1).$$

ROC curves

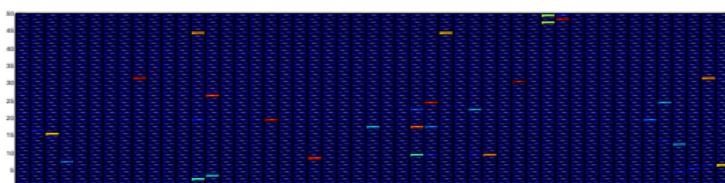
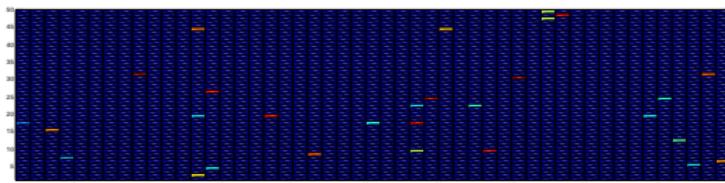
- Below: ROC curves for sparsity/number of targets $s = 100$ at Rayleigh resolution.



Gridding error

- **Question:** what happens for finer grids? Set $SRF \in \mathbb{N}_{>1}$ (**super-resolution factor**), choose meshsize $d_0 = \frac{1}{SRF} \frac{\lambda z_0}{A}$.

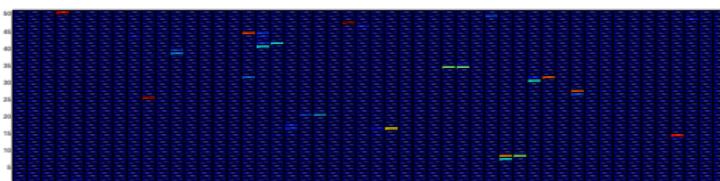
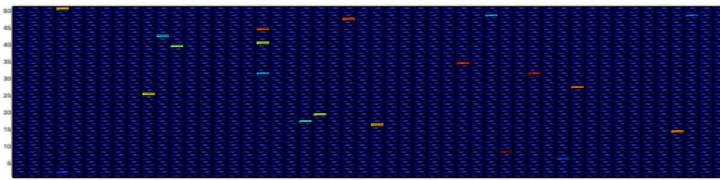
SRF = 4, 30 targets, SNR = 27dB



Gridding error

- **Question:** what happens for finer grids? Set $SRF \in \mathbb{N}_{>1}$ (**super-resolution factor**), choose meshsize $d_0 = \frac{1}{SRF} \frac{\lambda z_0}{A}$.

SRF = 8, 20 targets, SNR = 27dB



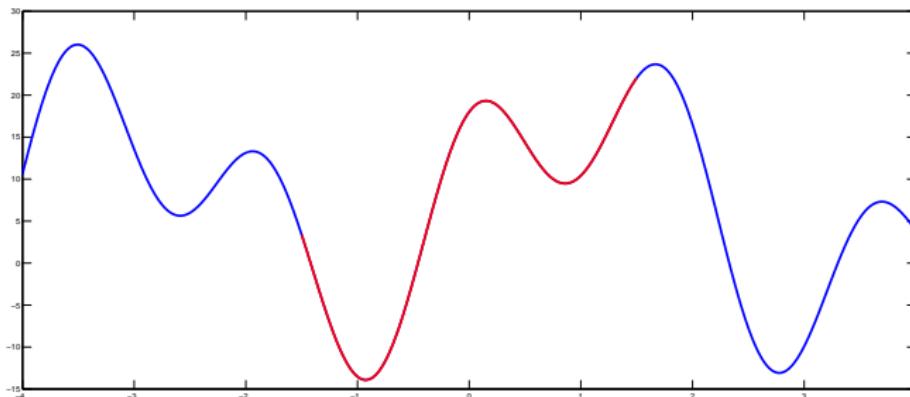
- Antenna elements $a_1, \dots, a_n \in [-A/2, A/2]^2$ on an array $[-A/2, A/2]^2$ arranged on equidistant grid of meshsize δ_0 ;
- $a_j = (k_1\delta_0, k_2\delta_0)$ with $k_1, k_2 \in \{-A/(2\delta_0), \dots, A/(2\delta_0)\}$.
- **Now:** Measure Fourier coefficients $y_{km} = (\mathcal{F}x)_{km} \in \mathbb{C}$ at **deterministic** frequencies $1/(\lambda z_0)(a_k + a_m)$.
 \Rightarrow Measure spectrum of scene $x = \sum_{\ell=1}^s x_\ell \delta_{p_\ell}$ in target domain $[-L, L]^2 \times \{z_0\}$ up to cut-off

$$f_c := A/(\lambda z_0).$$

\Rightarrow Recover x from **low-pass** information $y = \mathcal{F}x \in \mathbb{C}^{2n+1}$ amounts to **spectrum extrapolation**.

Low-pass sensing

Super-Resolution \cong Spectrum extrapolation



A rigorous SR-result

In practice: Discretize $[-L, L]^2 \times \{z_0\}$ into grid $\{p_j\}_{j=1,\dots,N}$.

$x \in \mathbb{C}^N$, $N = SRF \cdot n$, measure Fourier coefficients

$y = F_n x \in \mathbb{C}^{2n+1}$, $n = A/(\lambda z_0)$. Set $T := \{j \in [N] | x_j \neq 0\}$,

$$SRF := N/n, \quad d_{\min} := \min_{j \neq k \in T} \|p_j - p_k\|_\infty.$$

Theorem (Candès, Fernandez-Granda)

Suppose measure $y = F_n x + z$ with $\|F_n^* z\|_1/N \leq \eta$. Let $d_{\min} \geq 2.38\lambda z_0/A$. Then solution \hat{x} to

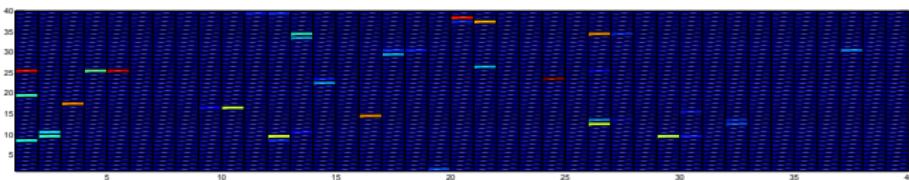
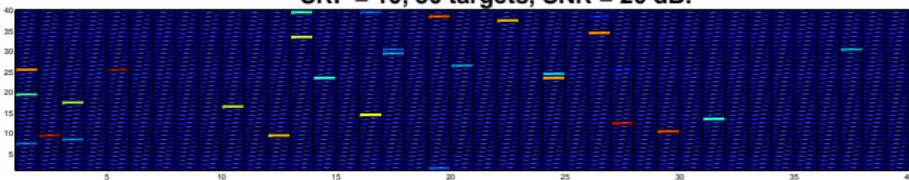
$$\arg \min_{\tilde{x} \in \mathbb{C}^N} \|\tilde{x}\|_1 \text{ subject to } \|F_n^*(F_n \tilde{x} - y)\|_1 / N \leq \eta$$

satisfies

$$\|\hat{x} - x\|_1 \leq \text{Const.} SRF^2 \eta.$$

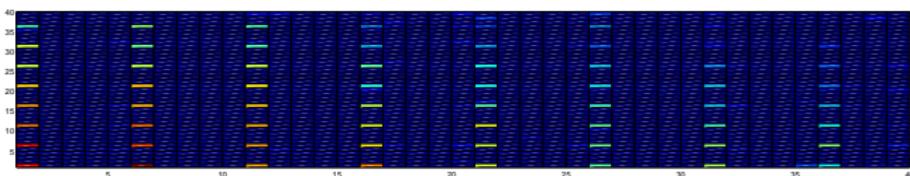
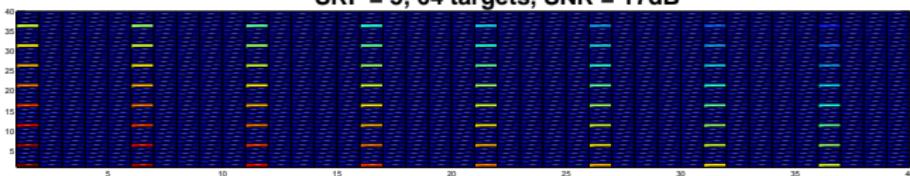
Numerical examples

SRF = 10, 30 targets, SNR = 20 dB.



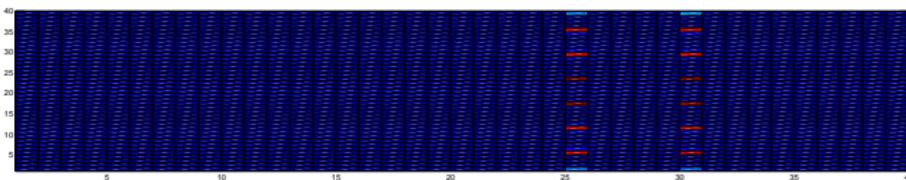
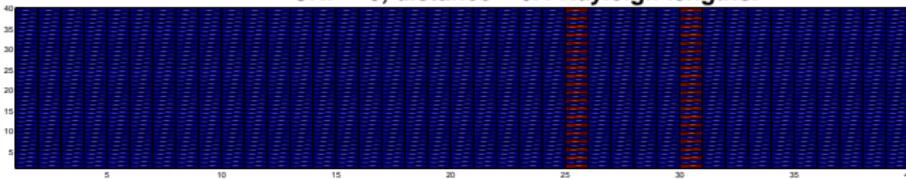
Numerical examples

SRF = 5, 64 targets, SNR = 17dB



Numerical examples

SRF = 5, distance = 0.4 Rayleigh lengths.



Thank you for your attention!