An Analytical Study of Sparse Recovery Algorithms in Presence of an Off-Grid Source

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Direction of arrival (DOA) estimation has been an active field of research for many decades. If the field is modeled as a superposition of a few planar wavefronts, the DOA estimation problem can be expressed as a sparse recovery problem and the Compressed Sensing (CS) framework can be applied. Many powerful CS-based DOA estimation algorithms have been proposed in recent years.

However, they all face one common problem. Although, the model is sparse in a continuous angular domain, to apply the CS framework we need to construct a finite dictionary by sampling this domain with a predefined sampling grid. Therefore, the target locations are almost surely not located exactly on a subset of these grid points.

Early solutions to this problem include adaptively refining the grid around the candidate targets found with an initial, mismatched grid [1]. Recent papers try to model the mismatch error explicitly and fit it to the observed data either statistically [2] or by interpolating between grid points [3].

In this paper we take an analytical approach to investigate the effect of recovering the spectrum of a source not contained in the dictionary. Unlike earlier works on the sensitivity of compressed sensing to basis mismatch [4] that have provided a quantitative analysis of the approximation error, we focus on the shape of the resulting spectrum, considering one target source for simplicity. We show that the recovered spectrum is not sparse but it can be well approximated by the closest two dictionary atoms on the grid and their coefficients can be exploited to estimate the grid offset.

Let $\boldsymbol{a}(\mu) \in \mathbb{C}^{M\times 1}$ denote the array manifold for an M-element uniform linear array. We construct a dictionary $\boldsymbol{A} \in \mathbb{C}^{M\times N}$ by sampling μ with a uniform grid chosen as $\mu_i = \Delta \cdot i$, $i = 0, 1, \ldots, N-1$, where $\Delta = \frac{2\pi}{N}$. The true location of the target source is given by $\mu_{\text{true}} = \mu_L + \epsilon$, where $0 \leq \epsilon < \Delta$ so that L and $L+1$ are the closest two grid points to μ_{true}. In the absence of noise, a sparse recovery algorithm returns coefficients α_i that represent $\boldsymbol{a}(\mu_{\text{true}})$ in the basis $\boldsymbol{a}(\mu_i)$. Our analysis for the distribution of α_i reveals some interesting insights. Firstly, the largest two coefficients are located at $i = L$ and $i = L+1$, i.e., the closest two grid points. The remaining $M-2$ coefficients are found in the vicinity of L and $L+1$ for greedy-type recovery algorithms such as Orthogonal Matching Pursuit (OMP) and farther away for ℓ_1-type algorithms such as Basis Pursuit (BP). Secondly, we can analytically compute the coefficients for the best approximation of $\boldsymbol{a}(\mu_{\text{true}})$ using only two coefficients L and $L+1$ and provide upper bounds for the approximation error showing that the approximation becomes very small for $N \gg M$. Thirdly, since the coefficients α_L and α_{L+1} depend on ϵ, we show that we can estimate ϵ via the relation $\epsilon \approx \hat{\epsilon} = \Delta \cdot \frac{\alpha_{L+1}}{\alpha_L + \alpha_{L+1}}$. The estimated ϵ can be used to adapt the dictionary or to guide grid refinement algorithms more efficiently.

To validate our observations we present an empirical result where we applied BP to estimate the angular spectrum for a single off-grid source (without additive noise) for $M = 8$, $N = 32$, and $\epsilon = 0.7 \cdot \Delta$ in Figure 1. The solid blue line indicates the true position μ_{true} and the red crosses show the recovered spectrum. We observe that most of the energy concentrates on the two adjacent grid points α_L and α_{L+1} and the remaining $M-2$ non-zero coefficients are located farther away. Based on the relative height of the dominant peaks we can estimate ϵ using the estimate $\hat{\epsilon}$ shown above. The dashed green line shows the corresponding estimated position. The relative estimation error is $\approx 9 \cdot 10^{-6}$.

In the full paper we also show reconstruction results for the OMP algorithm. The main difference is that the non-zero coefficients concentrate in the vicinity of the actual source location. Still, the estimation of the true location based on $\hat{\epsilon}$ works very well.

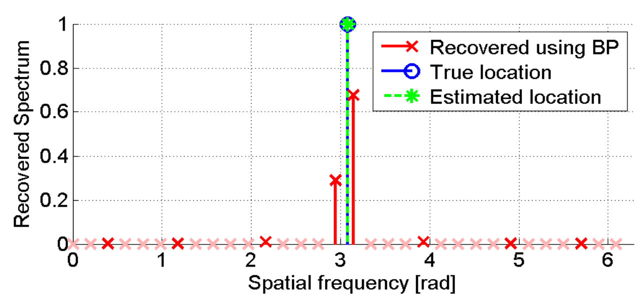

Fig. 1. Recovered spectrum using Basis Pursuit (BP)

References

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