On the choice of mixing sequences for SNR improvement in Modulated Wideband Converter

2nd International Workshop on Compressive Sensing applied to Radar

17 - 19 September 2013, Bonn, Germany

Anastasia Lavrenko, Florian Römer, Giovanni Del Galdo and Reiner S. Thomä

Institute for Information Technology, TU Ilmenau, Germany

Contact Information

- Prof. Giovanni Del Galdo

  Head of DVT Research Group, composed of
  Digital Broadcasting Research Laboratory,
  Ilmenau University of Technology
  Wireless Distribution Systems / Digital Broadcasting Group,
  Fraunhofer Institute for Integrated Circuits IIS

  office: +49 3677 69-4280
  mobile: +49 151 62 91 01 81
  e-mail: giovanni.delgaldo@tu-ilmenau.de
  giovanni.delgaldo@iis.fraunhofer.de

- Prof. Reiner S. Thomä

  Head of the Electronic Measurement Research Laboratory
  (Fachgebiet Elektronische Messtechnik)
  Ilmenau University of Technology

  office: + 49 3677 69-2622
  mobile: + 49 3677 69-1113
  e-mail: reiner.thomae@tu-ilmenau.de

- M.Sc. Anastasia Lavrenko

  PhD Student in the *International Graduate School on Mobile Communications*
  Electronic Measurement Research Lab
  Ilmenau University of Technology

  office: +49 3677 69 1397
  e-mail: anastasia.lavrenko@tu-ilmenau.de

- Dr.-Ing. Florian Römer

  Digital Broadcasting Research Laboratory,
  Ilmenau University of Technology

  office: +49 3677 69 4286
  e-mail: florian.roemer@tu-ilmenau.de

Mailing address:
Ilmenau University of Technology,
Helmholzplatz 2
98693 Ilmenau, Germany

Introduction

Noiseless CS formulation

$$\boldsymbol{d} = \boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{s} = \boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{B}\boldsymbol{x}$$

$$\boldsymbol{d} = \boldsymbol{\Phi}^{\mathrm{T}}(\boldsymbol{s} + \boldsymbol{w}) + \cancel{\boldsymbol{e}} = \boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{A}(\boldsymbol{x} + \boldsymbol{n}) + \cancel{\boldsymbol{e}}$$

'Signal' noise

Measurement noise

Input SNR

$$\eta_{\mathrm{I}} = \frac{\|\boldsymbol{x}\|_2^{\,2}}{\mathrm{E}\{\|\boldsymbol{n}|\boldsymbol{\lambda}\|_2^{\,2}\}}$$

Output SNR

$$\eta_{\mathrm{O}} = \frac{\|\boldsymbol{B}\boldsymbol{x}\|_2^{\,2}}{\mathrm{E}\{\|\boldsymbol{B}\boldsymbol{n}\|_2^{\,2}\}}$$

Recovered SNR

$$\eta_{\mathrm{R}} = \frac{\|\boldsymbol{x}\|_2^{\,2}}{\mathrm{E}\{\|\hat{\boldsymbol{x}} - \boldsymbol{x}\|_2^{\,2}\}}$$

$$\eta_{\mathrm{I}} \Longrightarrow \eta_{\mathrm{O}} \Longrightarrow \eta_{\mathrm{R}}$$

Outline

- Modulated Wideband Converter
- Noise Folding in the Modulated Wideband Converter
- Frequency-punctured Mixing Sequences
- Some Numerical Results
- SNR - Coherence
- Conclusions

MWC for blind wideband signal acquisition

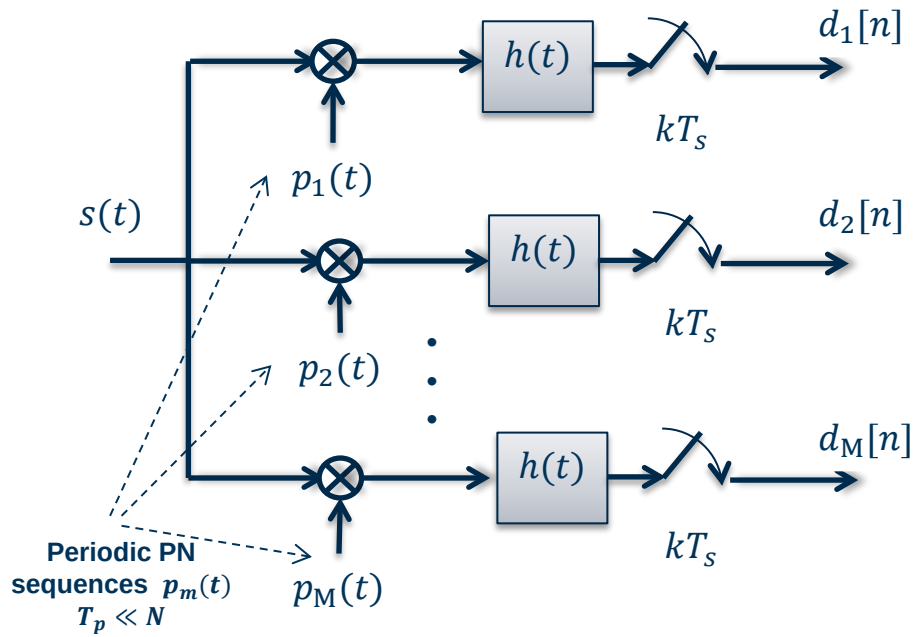
- Multiband signal
  - Multiple narrowband frequency channels with unknown frequency positions distributed over wide frequency band of interest W.
  - Limited number of active channels L_s so that $\sum_{i=1}^{L_s} B_i \ll W \;\; \Rrightarrow$ sparse in the frequency domain

Given by nature

$$\boldsymbol{d} = \boxed{\boldsymbol{\Phi}^{\mathrm{T}}}\boxed{\boldsymbol{A}\boldsymbol{x}}$$

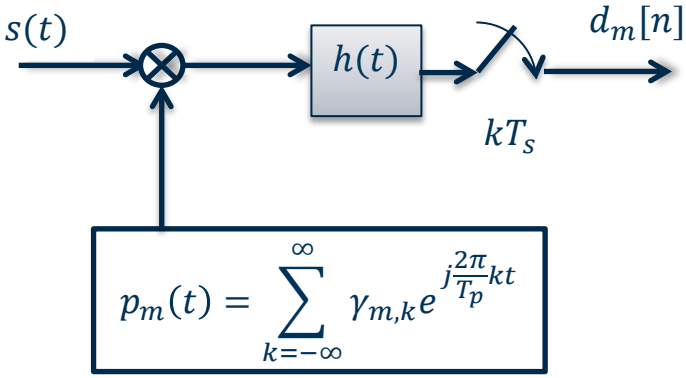
Implemented in hardware

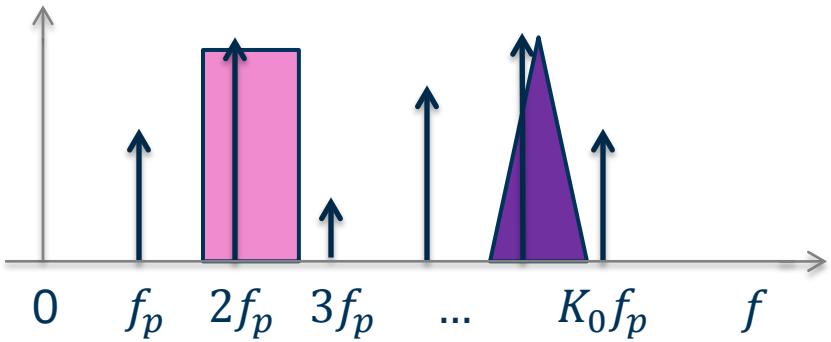
$$\boxed{\boldsymbol{d} = \boldsymbol{B}\boldsymbol{x} = \boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{s} = \boldsymbol{R}\boldsymbol{\Psi}\boldsymbol{A}\boldsymbol{s}}$$

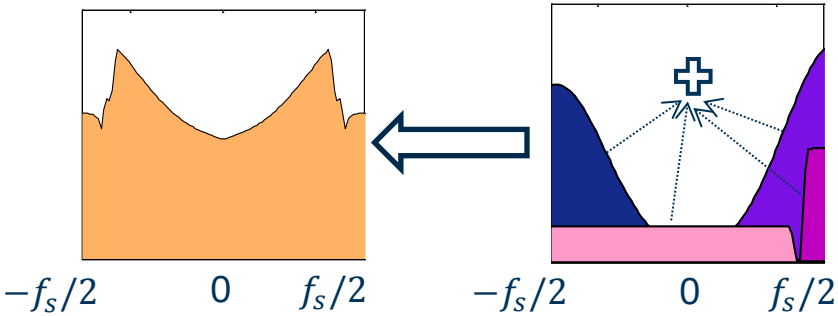


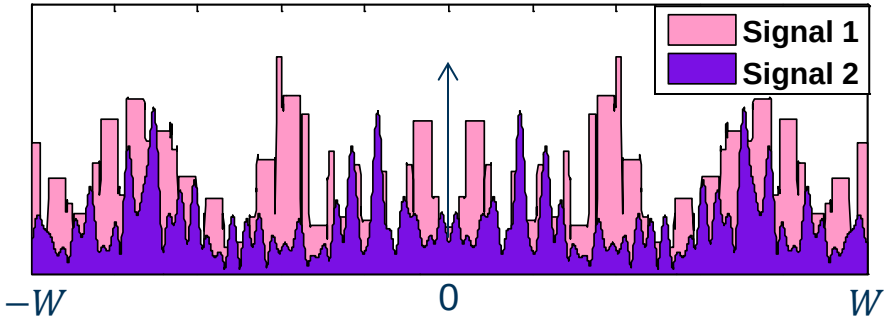
*Mishali, Moshe, and Yonina C. Eldar. "From theory to practice: Sub-Nyquist sampling of sparse wideband analog signals." *Selected Topics in Signal Processing, IEEE Journal of* 4.2 , 2010.

MWC in frequency domain analysis









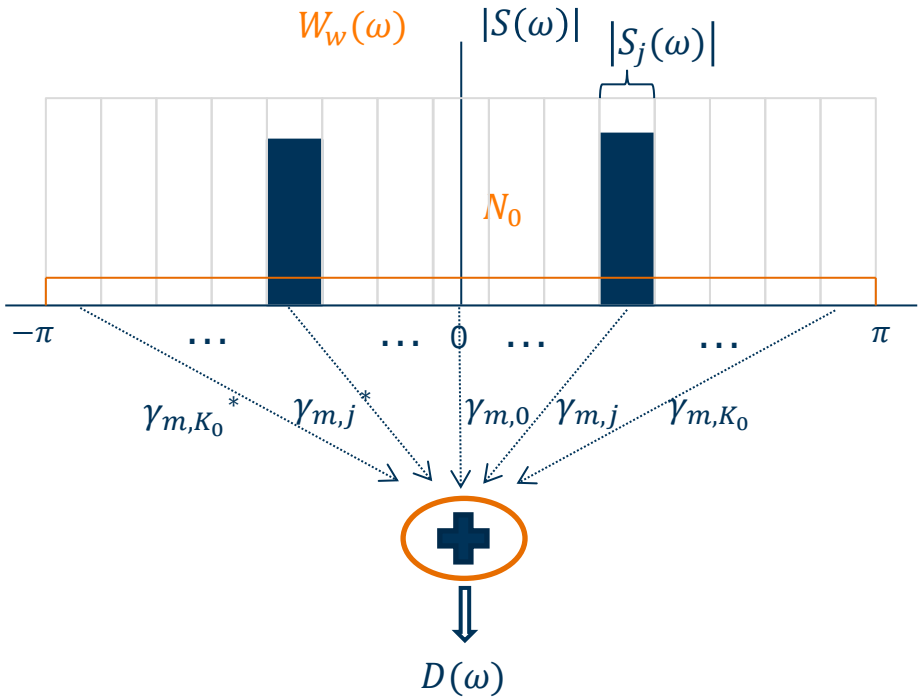
Noise folding in the MWC

Input SNR

$$\eta_{\mathrm{I}} = \frac{total\ input\ signal\ power}{noise\ power\ within\ occupied\ bands}$$

Output SNR per channel

$$\eta_{\mathrm{O},m} = \frac{total\ output\ signal\ power}{total\ ouput\ noise\ power}$$



Assumptions:

- $f_s = f_p$
- Sub-band bandwidths $B_j = \frac{2W}{2K_0+1} = \frac{2W}{K}$
- Sub-band central frequencies $f_{cj} = j \cdot {}^{f_p}/_2$
- Input noise is white with PSD N_0
- $\gamma_{m,k} \sim \mathcal{CN}(0, 1/\sqrt{K})$

$$\eta_{\mathrm{I}} = \frac{\sum_{j\in\mathcal{K}_s} P_{S_j}}{N_0 2W/K} \qquad \eta_{\mathrm{O},m} \leq \frac{\sum_{j\in\mathcal{K}_s}\left(\left|\gamma_{m,j}\right|^2 \cdot P_{S_j}\right)}{\frac{N_0 2W}{K} \cdot \sum_k \left|\gamma_{m,k}\right|^2}$$

$\mathrm{E}\left\{\eta_{\mathrm{O},m}\right\} = \frac{1}{K}\eta_I$,where $\frac{1}{K}$ is sampling rate reduction per channel

Frequency-punctured mixing sequences

Let's build $\boldsymbol{\gamma}_m$ so that

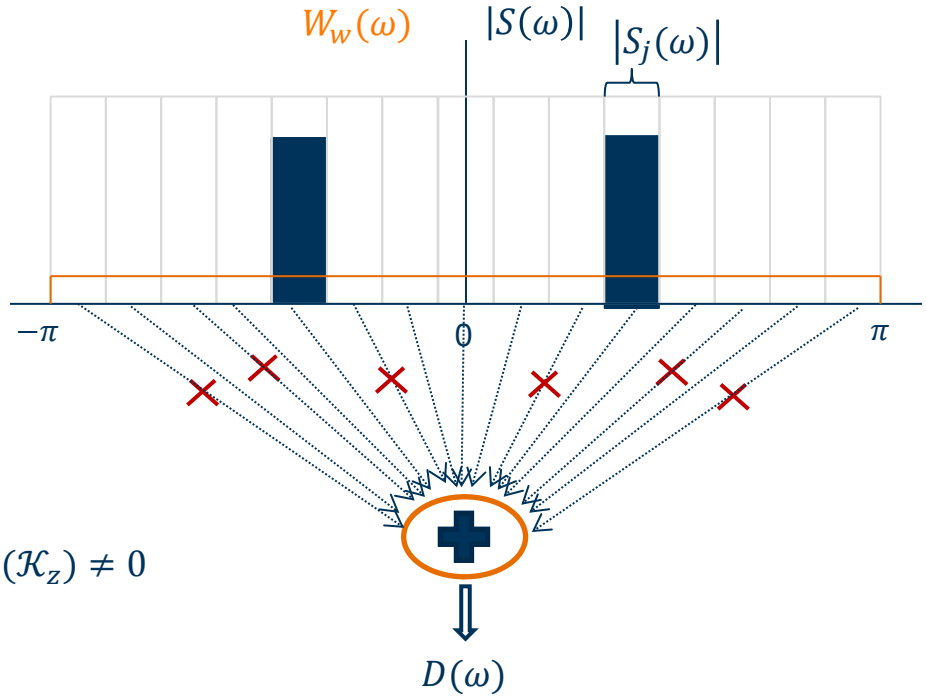
$$D_m(\omega) = \sum_{k \in \mathcal{K}_m} \gamma_{m,k} \cdot S_k(\omega),$$

where $\mathcal{K}_m \subseteq \mathcal{K}: \{-K_0, \dots, 0, \dots, K_0\}$ defines the number $\hat{K} \leq K$ of subbands to take into account

$$\boxed{\begin{aligned} &\gamma_{m,k} = 0, \text{for } k \notin \mathcal{K}_m \text{ and} \\ &\mathrm{E}\{|\gamma_{m,k}|\} = \frac{1}{\sqrt{\hat{K}}}, \text{ for } k \in \mathcal{K}_m \end{aligned}}$$

$$\mathrm{E}\left\{\eta_{\mathrm{O},m}\right\} = \mathrm{E}\left\{\frac{\sum_{j \in \mathcal{K}_s \cap \mathcal{K}_m} \left(|\gamma_{m,j}|^2 \cdot P_{S_j}\right)}{\frac{N_0 W}{K} \cdot \sum_{k \in \mathcal{K}_m} |\gamma_{m,k}|^2}\right\}$$

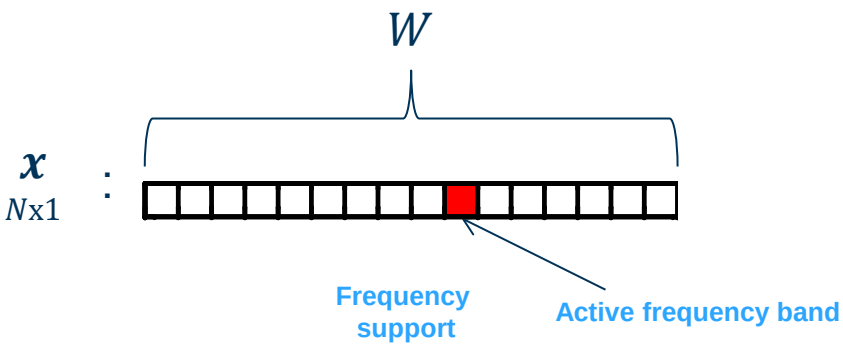
$$\mathrm{E}\left\{\eta_{\mathrm{O},m}\right\} = \begin{cases} \frac{1}{\hat{K}} \cdot \eta_I, \text{when } \mathcal{K}_s \subseteq \mathcal{K}_m \\ \frac{1}{\hat{K}} \cdot \beta \cdot \eta_I, \text{when } \mathcal{K}_s \cap \mathcal{K}_m = \mathcal{K}_z, \mathrm{n}(\mathcal{K}_z) \neq 0 \\ 0, \mathcal{K}_s \cap \mathcal{K}_m = \emptyset \end{cases}$$



Simplified signal model

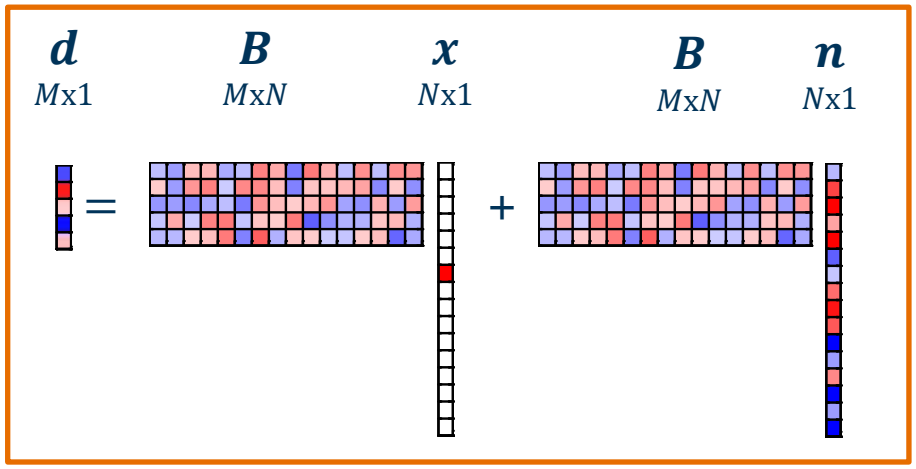
Simplified frequency domain signal model:

- $\boldsymbol{x} \in \mathbb{C}^N, \mathrm{card}(supp\{|\boldsymbol{x}|\}) = L_s = 1\ ,\ \|\boldsymbol{x}\|^2 = 1$
- $\boldsymbol{n} \in \mathbb{C}^N, n_k \sim \mathcal{CN}(0, N_0)$
- $\gamma_{m,k} \sim \mathcal{CN}(0,1),\ k = 1, \dots N,\ \mathrm{E}\{\|\boldsymbol{\gamma}_m\|^2\} = 1$

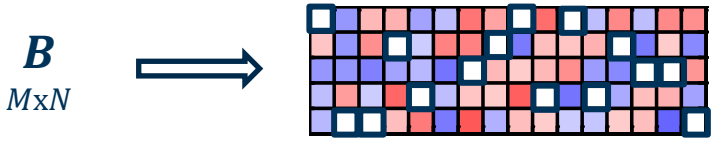


$$\boldsymbol{d} = \boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{r} = \boldsymbol{\Phi}^{\mathrm{T}}(\boldsymbol{s} + \boldsymbol{w}) = \boldsymbol{B}(\boldsymbol{x} + \boldsymbol{n})$$

$$\boldsymbol{B} = \boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{A} = \begin{bmatrix} \gamma_{1,1} & \cdots & \gamma_{1,N} \\ \vdots & \ddots & \vdots \\ \gamma_{m,1} & \cdots & \gamma_{m,N} \end{bmatrix} \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$

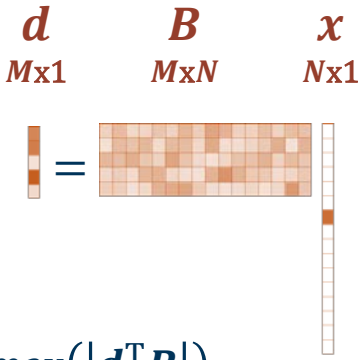


Introducing $K - \hat{K}$ zeros



so that $\forall k. \exists \gamma_{m,k}:\ \gamma_{m,k} \neq 0$

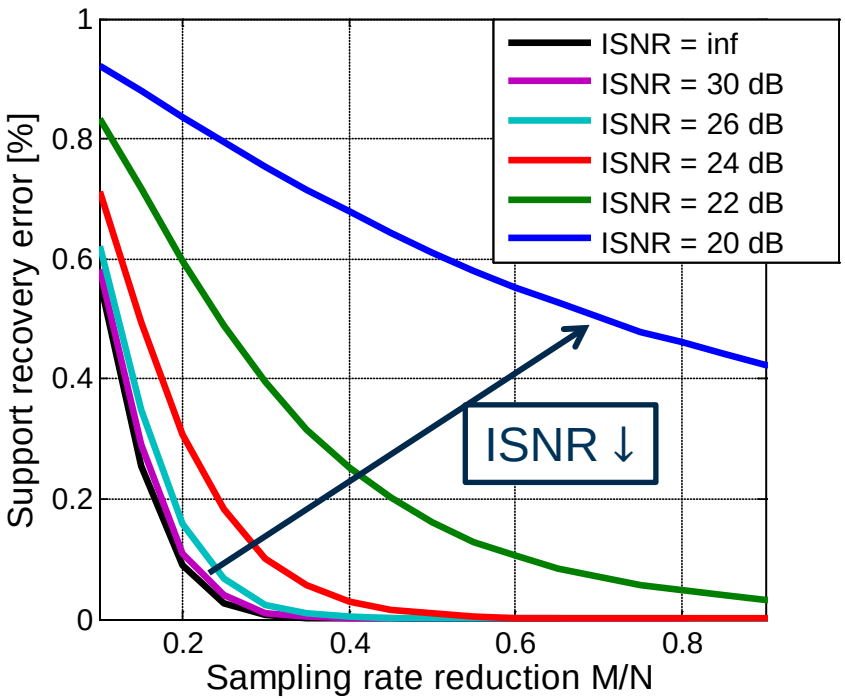
Numerical results. Support recovery

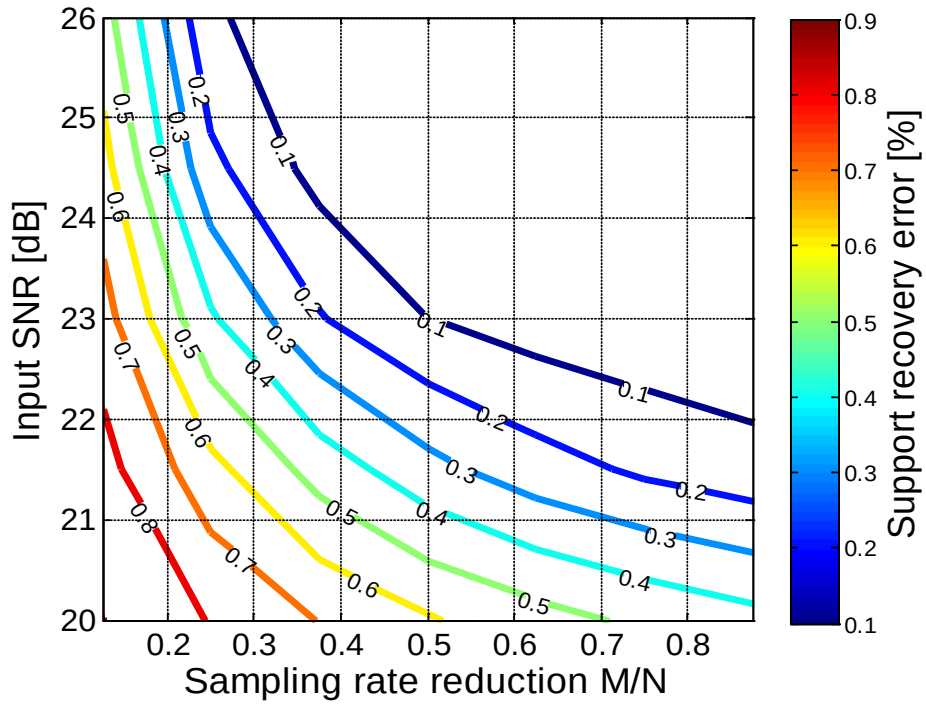


Support recovery was based on the Basic Pursuit approach

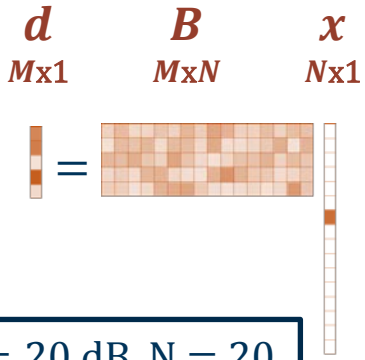
$$\min_{x} \|\boldsymbol{x}\|_1 \text{ subject to } \|\boldsymbol{d} - \boldsymbol{B}\boldsymbol{x}\|_2^2 \leq \varepsilon \qquad \mathrm{card}(\boldsymbol{x}) = 1 \rightarrow supp(\boldsymbol{x}) = \max\left(\left|\boldsymbol{d}^{\mathrm{T}}\boldsymbol{B}\right|\right)$$

$\frac{\hat{K}}{K} = 1$, Sparsity $\frac{L_s}{N} = 0.025$





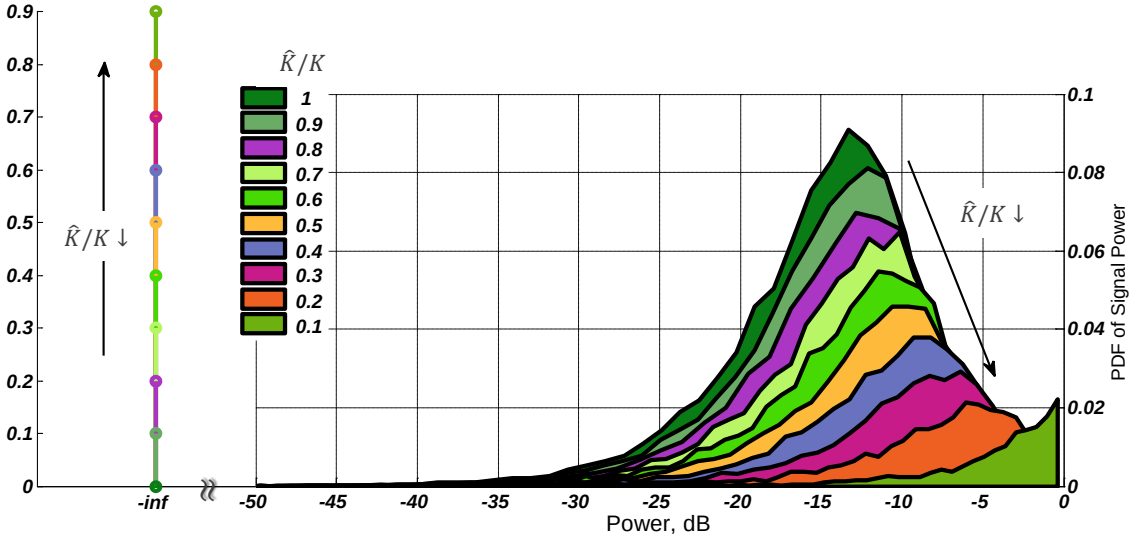
Numerical results. Punctured γ_m SNR per channel



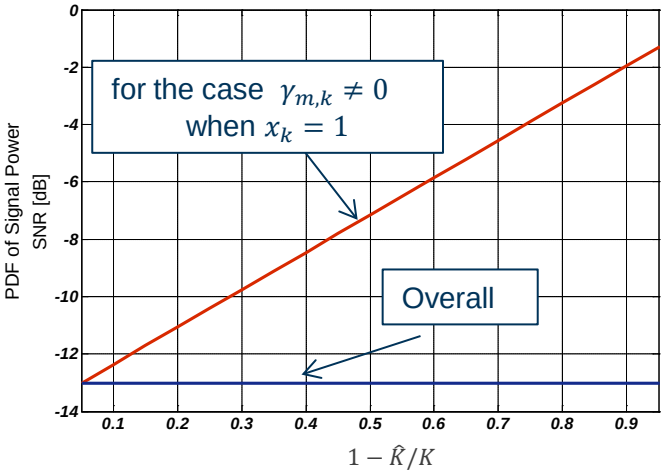
$$\eta_{\mathrm{I}} = \frac{\|\boldsymbol{x}\|^2}{\mathrm{E}\{\|n_k\|^2\}} = \frac{1}{N_0}, \qquad \eta_{\mathrm{O},m} = \frac{\mathrm{E}\left\{\left\|\boldsymbol{\gamma}_m^{\mathrm{T}}\boldsymbol{x}\right\|^2\right\}}{\mathrm{E}\left\{\left\|\boldsymbol{\gamma}_m^{\mathrm{T}}\boldsymbol{n}\right\|^2\right\}}$$

Input SNR $\eta_{\mathrm{I}} = 20$ dB, $\mathrm{N} = 20$

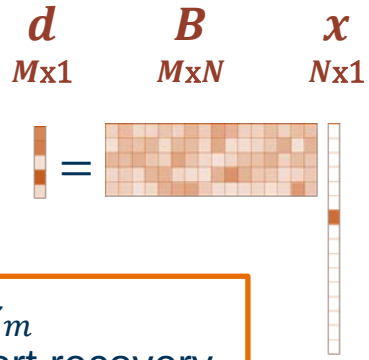
Probability density function of the output signal power



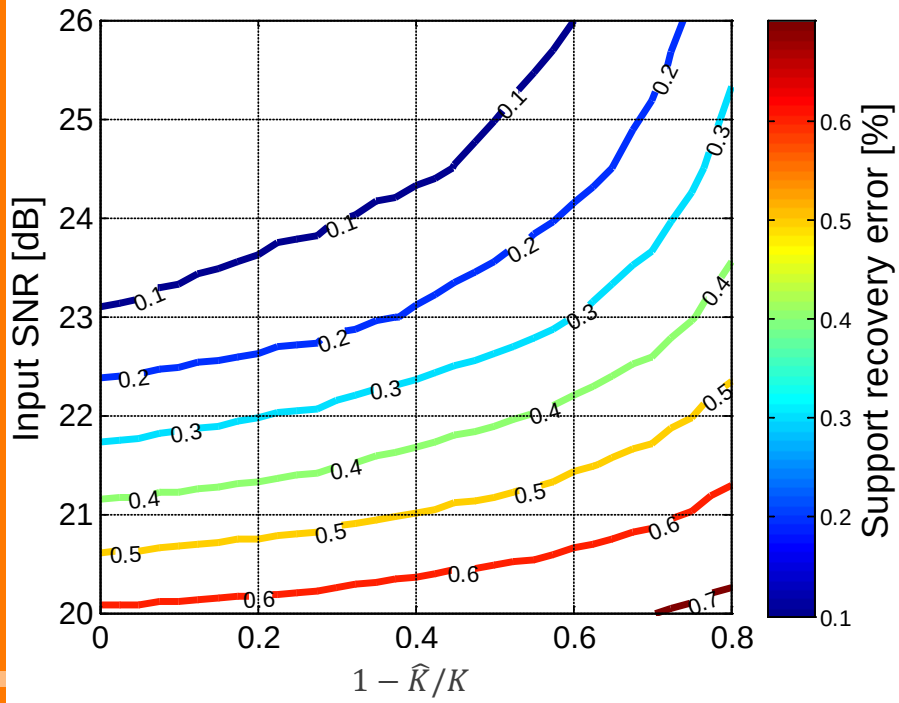
Expected output SNR



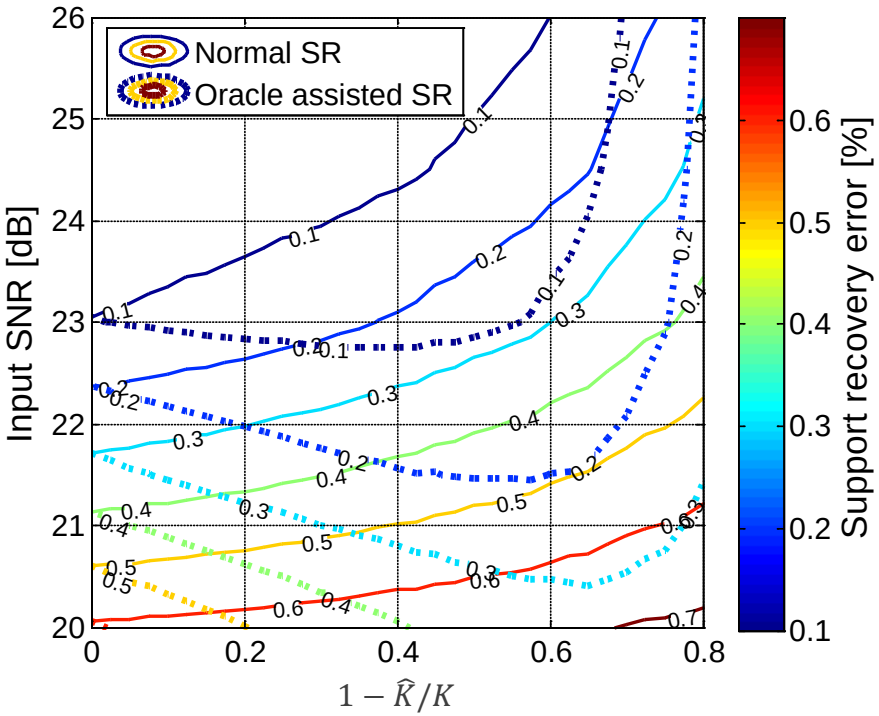
Numerical results. Punctured γ_m Support recovery



Punctured γ_m

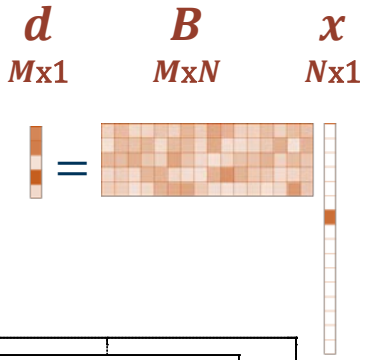


Punctured γ_m
Oracle assisted support recovery



Sampling rate reduction $\frac{M}{N} = 0.5$, Sparsity $\frac{L_s}{N} = 0.025$

Numerical results. Punctured γ_m
Support recovery and coherence

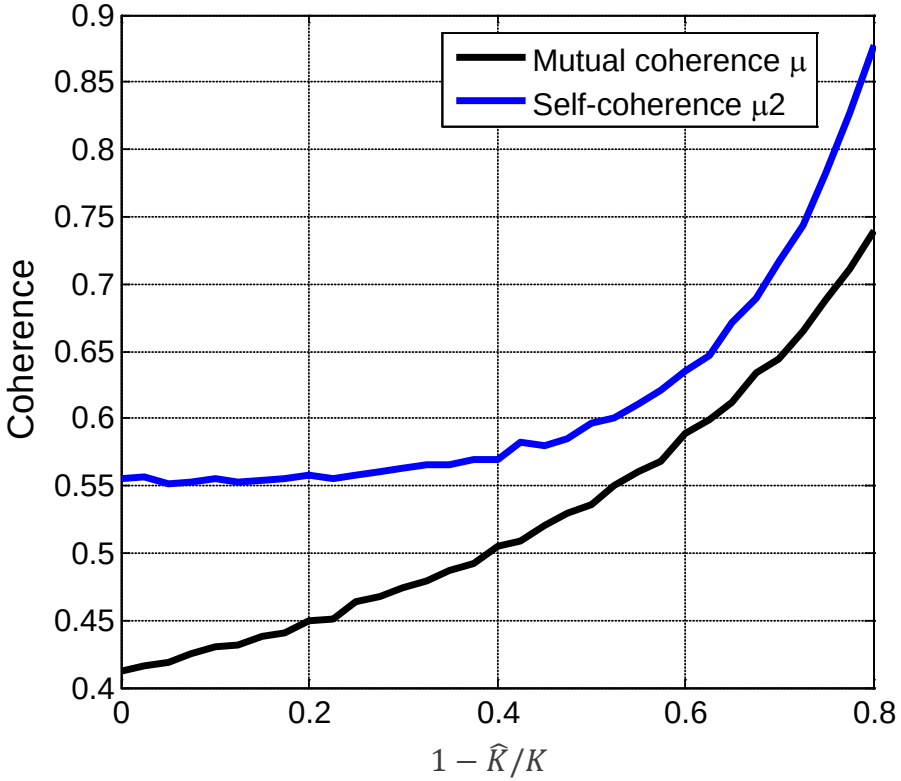


Mutual Coherence μ between $\boldsymbol{\Phi}$ and $\boldsymbol{A}$

$$\mu = \max_{i,j} \frac{\left|\langle \boldsymbol{\varphi}_i, \boldsymbol{a}_j \rangle\right|}{\|\boldsymbol{\varphi}_i\| \|\boldsymbol{a}_j\|}$$

Self-Coherence μ_2 of matrix $\boldsymbol{B}$

$$\mu_2 = \max_{\substack{i,j \\ i \neq j}} \frac{\left|\langle \boldsymbol{b}_i, \boldsymbol{b}_j \rangle\right|}{\|\boldsymbol{b}_i\| \|\boldsymbol{b}_j\|}$$

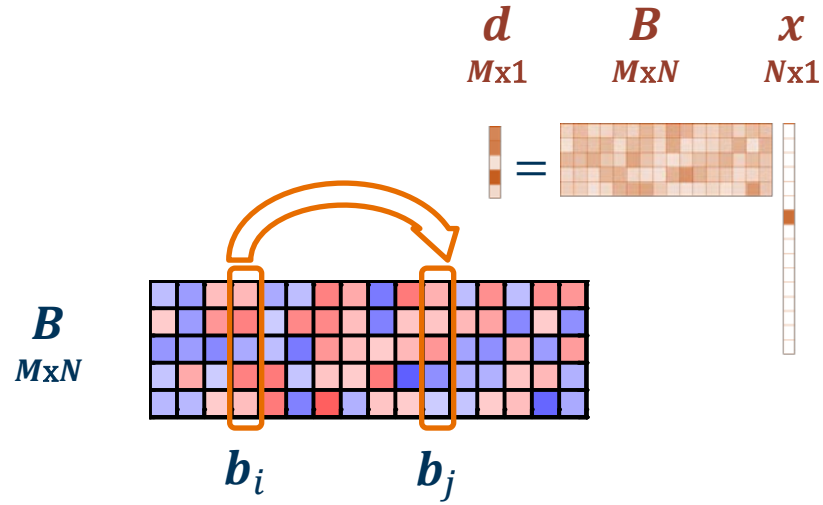


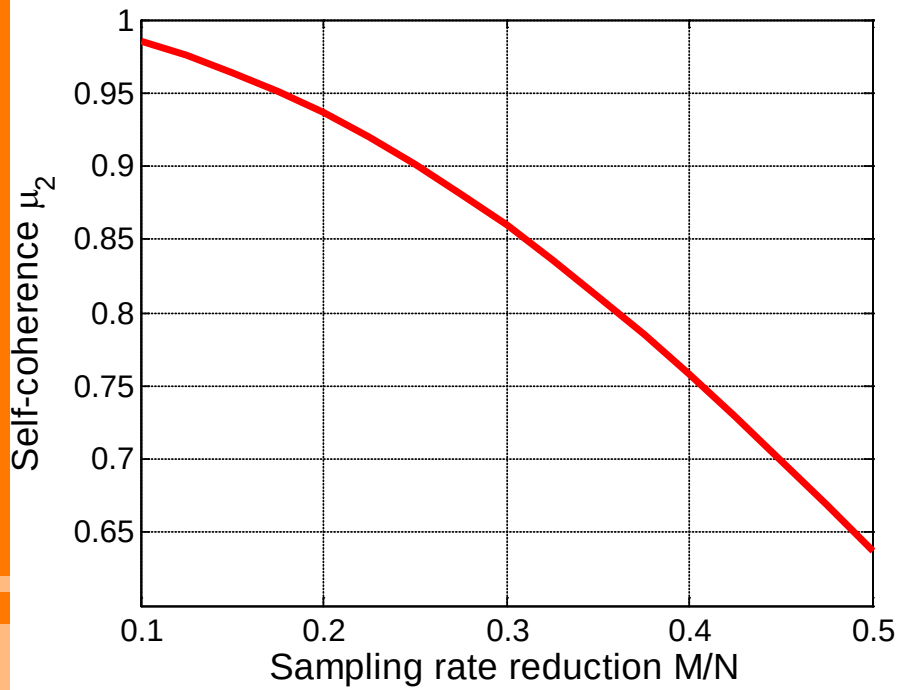
Sampling rate reduction $\frac{M}{N} = 0.5$, Sparsity $\frac{L_S}{N} = 0.025$

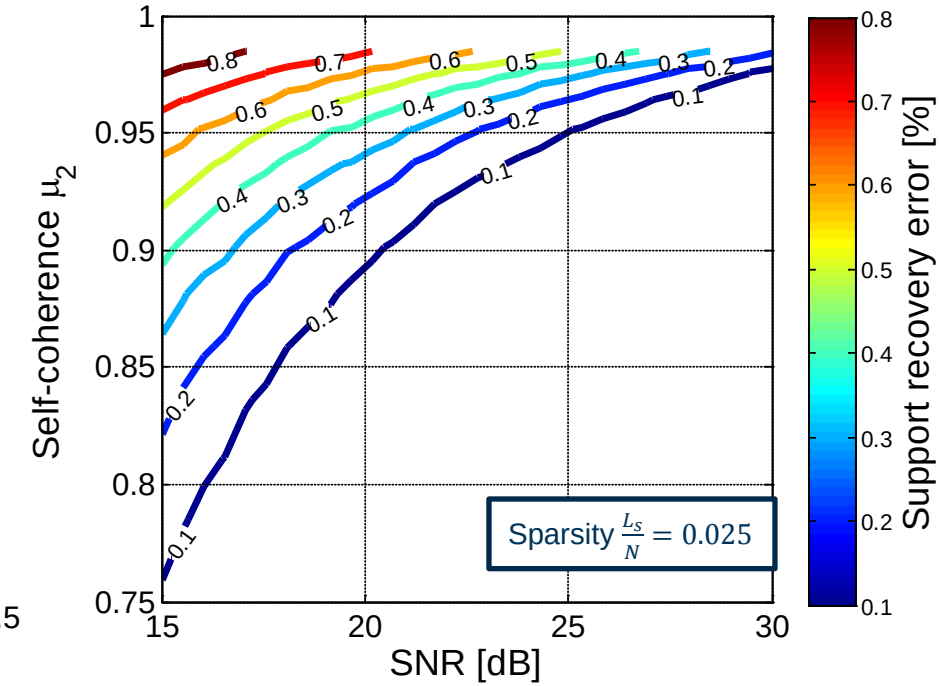
SNR and self-coherence

Self-coherence μ_2 of matrix $\boldsymbol{B}$

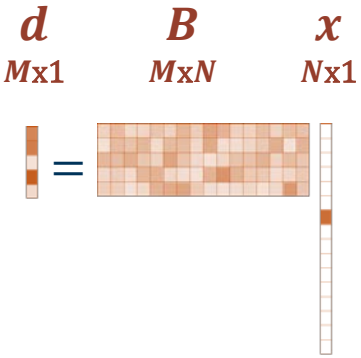
$$\mu_2 = \max_{\substack{i,j \\ i \neq j}} \frac{|\langle \boldsymbol{b}_i, \boldsymbol{b}_j \rangle|}{\|\boldsymbol{b}_i\| \|\boldsymbol{b}_j\|}$$







SNR and self-coherence



$$\boldsymbol{d}^{\mathrm{T}}\boldsymbol{B} = (\boldsymbol{B}\boldsymbol{x})^{\mathrm{T}}\boldsymbol{B} + (\boldsymbol{B}\boldsymbol{n})^{\mathrm{T}}\boldsymbol{B} = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B} + \boldsymbol{n}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B}$$

Probability of error due to noise

Probability of confusing i^{th} row of $\boldsymbol{B}$ with j^{th}

$$P_{\mathrm{E}}(i) = \left[\left|1 + \boldsymbol{n}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{b}_j\right| \leq \left|\beta_{ij} + \boldsymbol{n}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{b}_i\right|\right],$$

where $\beta_{ij} = \boldsymbol{b}_j^{\mathrm{T}}\boldsymbol{b}_i$

If $\boldsymbol{n}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}$ white noise with variance σ^2

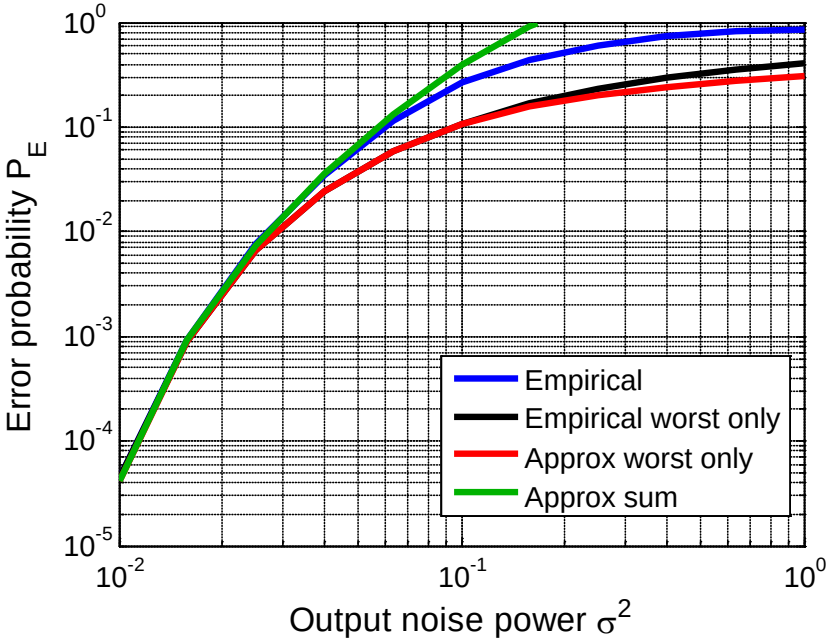
$$P_{\mathrm{E}}(i) \approx Q\left(-\sqrt{\frac{1-\beta_{ij}}{2\sigma^2}}\right) \cdot Q\left(\frac{\beta_{ij}}{\sqrt{\sigma^2}}\right) + Q\left(-\sqrt{\frac{1+\beta_{ij}}{2\sigma^2}}\right) \cdot Q\left(-\frac{\beta_{ij}}{\sqrt{\sigma^2}}\right)$$

Total probability of mistake

$$P_{\mathrm{E}} = \left[\left|1 + \boldsymbol{n}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{b}_j\right| \leq \max_i \left|\beta_{ij} + \boldsymbol{n}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{b}_i\right|\right]$$

Upper bound $P_E \leq \sum_i P_E(i)$

Lower bound $P_E(i_{worst}) \leq P_E$



Conclusions

- The effect of noise folding significantly degrades recovery performance.
- Noise mitigation capabilities via measurement kernel manipulations are limited by the coherence properties of the resulting kernel.
- The influence of the noise added to the signal prior measurement on the support recovery has been studied numerically revealing the relation between input SNR, number of channels, mutual and self-coherence.

Thank you for your attention!

Anastasia Lavrenko
Electronic Measurement lab
International Graduate School on Mobile Communications
TU Ilmenau, Germany
anastasia.lavrenko@tu-ilmenau.de