Compressive Spectrum Sensing with Spectral Priors for Cognitive Radar

UC San Diego
JACOBS SCHOOL OF ENGINEERING
Electrical and Computer Engineering

{ Ali Koochakzadeh, Heng Qiao and Piya Pal }
Department of Electrical and Computer Engineering, University of California, San Diego*

Introduction

WSS signal received by cognitive radar:

$$y(t) = \sum_{k=1}^{K} c_k e^{\jmath 2\pi f_k t/T} + n(t)$$

- c_k's: zero mean uncorrelated random variables
- $n(t)$: a WSS signal (possibly colored noise).
- $1/T$: the Nyquist sampling rate.
- $f_1, \cdots, f_k$: frequency of interfering signals

Contributions:

- We propose a sub-Nyquist sampling technique: Coprime Sampling.
- We use atomic norm denoising to detect the frequencies.
- We use *spectral priors* to improve the frequency localization.

Coprime Sampling

Auto-correlation:

$$r_{yy}[k] = E(y(nT)y^*(nT-kT)) = \sum_{m=1}^{K} \sigma_m^2 e^{\jmath k 2\pi f_m} + r_{nn}[k]$$

- $\sigma_m^2 = E(|c_m|^2)$.
- $r_{nn}[k]$ is the autocorrelation of $n(t)$, and $|r_{nn}[k]| < \eta$.

Coprime Sampling: Sample $y(t)$ using two samplers with rates $1/MT, 1/NT$ Hz

$$x_M[n] = y(MnT),$$
$$x_N[n] = y(NnT)$$

Lemma 1 [2] *$\forall\ 0 \le k \le MN-1$, $\exists\ 0 \le n_1 \le 2N-1$, $0 \le n_2 \le M-1$, so that $k = Mn_1 - Nn_2$.*

Estimating the Auto-Correlation:

1. Find $0 \le n_1 \le 2N-1, 0 \le n_2 \le M-1$, so that $k = Mn_1 - Nn_2$.
2. Compute $\tilde{r}_{yy}[k] = \frac{1}{L}\sum_{l=0}^{L-1} x_M[n_1 + Nl] x_N^*[n_2 + Ml]$
3. Estimate $\tilde{r}_{yy}[-k]$ using $(-n_1, -n_2)$, and refine as $\hat{r}_{yy}[k] = 0.5 \times (\tilde{r}_{yy}[k] + \tilde{r}_{yy}^*[-k])$

Restricted Spectral Priors

- **Unconstrained Atomic Set:** $\mathcal{A} \triangleq \{\mathbf{a}(f), f \in [0,1)\}$ where $\mathbf{a}(f) = [1, e^{\jmath 2\pi f}, e^{\jmath 2\pi 2f}, \cdots e^{\jmath 2\pi (P-1)f}]^T$
- The atomic norm of $\mathbf{x} \in \mathbb{C}^P$: $\|\mathbf{x}\|_{\mathcal{A}} = \inf\{t > 0 | \mathbf{x} \in t\text{conv}(\mathcal{A})\}$

- **Probabilistic Priors**: Band $\mathcal{F}_i \triangleq [f_{i1}, f_{i2})$ contains interfering frequencies w. p. α_i

$$\mathcal{A}_w = \{\mathbf{a}(f)w(f), f \in [0,1)\}$$
$$w(f) = \alpha_i, \text{for } f \in \mathcal{F}_i.$$

- **Block Priors**: Only the bands $\mathcal{F} = \bigcup_{i=1}^{I} \mathcal{F}_i$ are occupied. The corresponding atomic set:

$$\mathcal{A}_p = \{\mathbf{a}(f), f \in \mathcal{F}\}$$

Primal Problem: $\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{r} - \mathbf{x}\|_2^2 + \lambda \|\mathbf{x}\|_{\bar{\mathcal{A}}}$ where $\bar{\mathcal{A}}$ is either $\mathcal{A}_w$ or $\mathcal{A}_p$, and $\mathbf{r} = [\hat{r}_{yy}[0], \cdots, \hat{r}_{yy}[P-1]]^T$, with $P \le MN$

Dual problem for Probabilistic Priors:

$$\max_{\mathbf{z}} \frac{1}{2}\left(\|\mathbf{r}\|^2 - \|\mathbf{r} - \mathbf{z}\|^2\right)$$
$$|\langle \mathbf{a}(f), \mathbf{z}\rangle| \le \alpha_i \lambda,$$
$$f \in \mathcal{F}_i, i = 1, 2, \cdots, I$$

Dual problem for Block Priors:

$$\max_{\mathbf{z}} \frac{1}{2}\left(\|\mathbf{r}\|^2 - \|\mathbf{r} - \mathbf{z}\|^2\right)$$
$$|\langle \mathbf{a}(f), \mathbf{z}\rangle| \le \lambda,$$
$$f \in \mathcal{F}_i, i = 1, 2, \cdots, I$$

References

[1] K. V. Mishra, M. Cho, A. Kruger, and W. Xu, "Spectral super-resolution with prior knowledge", IEEE Trans. on Signal Proc. , vol. 63, no. 20, pp. 5342-5357, Oct. 2015.

[2] P. P. Vaidyanathan and Piya Pal, "Sparse Sensing With Co-Prime Samplers and Arrays" , IEEE Trans. on Signal Proc. , vol. 59, pp. 573-586, Feb. 2011.

* The authors were previously with University of Maryland, College Park.

Recasting the Dual Problems as SDP

Following [1], for $[f_L, f_H] \subset [0,1]$, define:

$$\alpha \triangleq \tan(\pi f_L), \beta \triangleq \tan(\pi f_H), d_0 \triangleq -\frac{\alpha\beta + 1}{2}, d_1 \triangleq \frac{1-\alpha\beta}{4} + \jmath\frac{\alpha+\beta}{4}$$

Furthermore, for Gram matrices $\mathbf{G}_1 \in \mathbb{C}^{P\times P}, \mathbf{G}_2 \in \mathbb{C}^{(P-1)\times(P-1)}$, let

$$\mathcal{L}_{k,f_L,f_H}(\mathbf{G}_1, \mathbf{G}_2) \triangleq \text{tr}(\Theta_k \mathbf{G}_1) + \text{tr}\left[(d_1\Theta_{k-1} + d_0\Theta_k + d_1^*\Theta_{k+1})\mathbf{G}_2\right]$$

Θ_k: ones on kth diagonal, and zeros elsewhere. ($k = 0$: the main diagonal, and $k > 0$: upper diagonals.)

Probabilistic Priors:

$$\underset{\substack{\mathbf{z} \\ \mathbf{G}_{11}, \mathbf{G}_{12}, \cdots, \mathbf{G}_{1I} \succeq 0 \\ \mathbf{G}_{21}, \mathbf{G}_{22}, \cdots, \mathbf{G}_{2I} \succeq 0}}{\text{maximize}} \quad \|\mathbf{r}\|^2 - \|\mathbf{r} - \mathbf{z}\|^2$$

subject to

$$\delta_{k_1} = \mathcal{L}_{k_1, f'_{L_1}, f'_{H_1}}(\mathbf{G}_{11}, \mathbf{G}_{21})$$
$$k_1 = 0, \cdots, P-1$$
$$\begin{bmatrix} \mathbf{G}_{11} & \frac{1}{\lambda\alpha_1}\tilde{\mathbf{z}}_{\tau_1} \\ \frac{1}{\lambda\alpha_1}\tilde{\mathbf{z}}_{\tau_1}^H & 1 \end{bmatrix} \succeq 0,$$
$$\vdots$$
$$\delta_{k_I} = \mathcal{L}_{k_I, f'_{L_I}, f'_{H_I}}(\mathbf{G}_{1I}, \mathbf{G}_{2I})$$
$$k_I = 0, \cdots, P-1$$
$$\begin{bmatrix} \mathbf{G}_{1I} & \frac{1}{\lambda\alpha_I}\tilde{\mathbf{z}}_{\tau_I} \\ \frac{1}{\lambda\alpha_I}\tilde{\mathbf{z}}_{\tau_I}^H & 1 \end{bmatrix} \succeq 0,$$

Block Priors:

$$\underset{\substack{\mathbf{z} \\ \mathbf{G}_{11}, \mathbf{G}_{12}, \cdots, \mathbf{G}_{1I} \succeq 0 \\ \mathbf{G}_{21}, \mathbf{G}_{22}, \cdots, \mathbf{G}_{2I} \succeq 0}}{\text{maximize}} \quad \|\mathbf{r}\|^2 - \|\mathbf{r} - \mathbf{z}\|^2$$

subject to

$$\delta_{k_1} = \mathcal{L}_{k_1, f_{L_1}, f_{H_1}}(\mathbf{G}_{11}, \mathbf{G}_{21})$$
$$k_1 = 0, \cdots, P-1$$
$$\begin{bmatrix} \mathbf{G}_{11} & \frac{1}{\lambda}\mathbf{z} \\ \frac{1}{\lambda}\mathbf{z}^H & 1 \end{bmatrix} \succeq 0,$$
$$\vdots$$
$$\delta_{k_I} = \mathcal{L}_{k_p, f_{L_I}, f_{H_I}}(\mathbf{G}_{1I}, \mathbf{G}_{2I})$$
$$k_I = 0, \cdots, P-1$$
$$\begin{bmatrix} \mathbf{G}_{1I} & \frac{1}{\lambda}\mathbf{z} \\ \frac{1}{\lambda}\mathbf{z}^H & 1 \end{bmatrix} \succeq 0,$$

Simulation Results

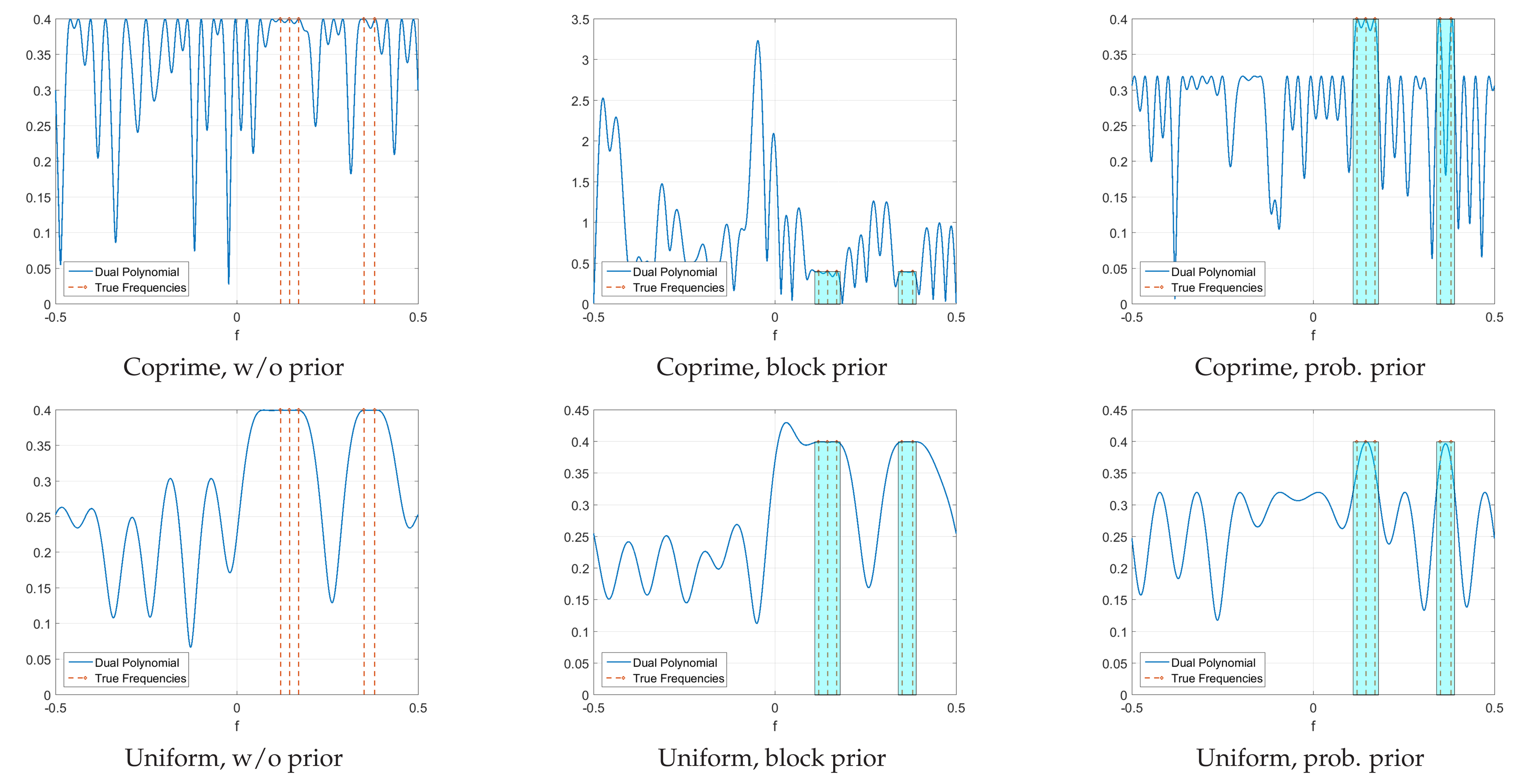

Figure 1. The dual polynomials corresponding to atomic norm denoising, using different samplers and spectral priors, $5 = K < \tilde{M} = 12$, $\tilde{M}$ indicates the block length, and K is the number of spectral lines. We use $L = 20$ blocks

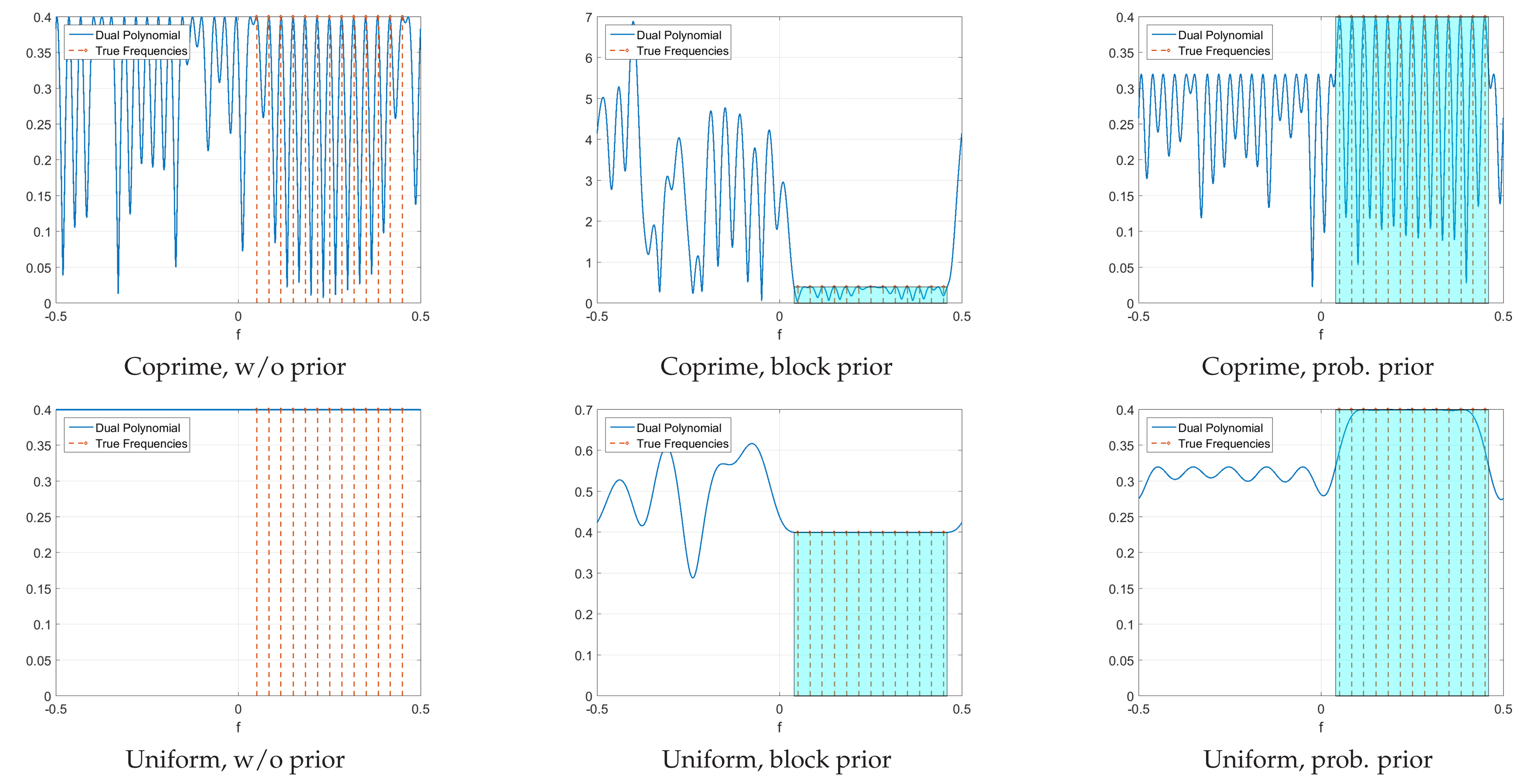

Figure 2. The dual polynomials corresponding to atomic norm denoising, using different samplers and spectral priors, $13 = K > \tilde{M} = 12$