COMPRESSIVE SENSING THEORY AND THE REAL WORLD

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- General about sensing
- (Interference and deviations from sparsity)
- Continuous recovery
- Random (?) projections
- Large scenes, large rawdata
- Non-linear sensing
- Higher level information retrieval
Compressive sensing techniques generally deal underdetermined systems of linear equations of the type \( y = A \boldsymbol{x} \):

\[
\begin{align*}
M & \quad = \\
\begin{bmatrix}
\text{ } & \text{ } & \text{ } \\
\end{bmatrix} \\
\begin{bmatrix}
\text{ } & \text{ } & \text{ } \\
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\boldsymbol{y} & \quad \begin{bmatrix}
\text{ } & \text{ } & \text{ } \\
\end{bmatrix} \\
\begin{bmatrix}
\text{ } & \text{ } & \text{ } \\
\end{bmatrix}
\end{align*}
\]

Under certain conditions on \( A \) there is an unique solution for \( S \)-sparse \( \boldsymbol{x} \).
APPLICATION OF CS TO REAL WORLD SENSORS (RADAR)

Questions to be asked

- What are the reasons to apply CS to a special radar task?
  - Sparing samples? (temporal, spatial, ...)
  - Better performance? (super resolution, image quality, additional information,...)
- Power budget preserved?
- Computing effort?
- Robustness, stability?
- Guarantee to fulfill the specs?
The concept of ‘compressible signals’, i.e. deviations from exact sparsity, and the robustness against interference have been treated adequately in the mathematical theory.
CHALLENGE II: CONTINUOUS RECOVERY

A lot of papers on the ‘off-grid problem’ or ‘continuous sparse recovery’ have been published. Nevertheless ...
CHALLENGE II: CONTINUOUS RECOVERY
Simulated scene with points between the grids

The scene model for CS is in principle discrete and finite, the real world is continuous. Prob(Position at a grid point) = 0.

Positions uniformly randomly chosen.

Top: reconstruction with matched filter.
Bottom: reconstruction with $\ell_1$-minimization.

The markers indicate the true values.

N = 500, M = 100, S = 5, noise = -30dB, partial Fourier matrix.
CHALLENGE II: CONTINUOUS RECOVERY

Hidden sidelobes

true positions
grid

shift to zero

accumulate and build histogram
Monte Carlo Simulation using \textit{spgl1} for sparse recovery

\begin{itemize}
  \item N = 128;
  \item M = 50;
  \item Noise -30 dB;
  \item Number of iterations = 10 000;
\end{itemize}
CHALLENGE II: CONTINUOUS RECOVERY

Model and approaches

\[ x(\psi) = \sum_{s=1}^{S} x_s \delta(\psi - \psi_s) \]

Measurements

\( \psi_1, \ldots, \psi_N \)
\( \hat{x}_1, \ldots, \hat{x}_N \)

CS processing

CS-Estimator of a spiky scene

\[
\begin{pmatrix}
\hat{x}_1 \\
\hat{\psi}_1
\end{pmatrix}, \ldots, 
\begin{pmatrix}
\hat{x}_N \\
\hat{\psi}_N
\end{pmatrix}
\]

Measurements

Post processing

\( \hat{\psi}_1, \ldots \)
\( \hat{x}_1, \ldots \)
\( \ldots, \hat{\psi}_N \)
\( \ldots, \hat{x}_N \)
CHALLENGE II: CONTINUOUS RECOVERY

How to measure the performance?

\[ x(\vartheta) = \sum_{s=1}^{S} x_s \delta(\vartheta - \vartheta_s) \]

‘spiky’ scene

\[ \hat{x}(\vartheta) = \sum_{s'=1}^{S'} \hat{x}_s \delta(\vartheta - \hat{\vartheta}_{s'}) \]

estimated scene (‘spiky’ too)

\( \hat{\vartheta}_{s'} \) may coincide with the grid points

How to measure the reconstruction quality?
CHALLENGE II: CONTINUOUS RECOVERY

A new performance measure (to be discussed!)

\[ x(\vartheta) = \sum_{s=1}^{S} x_s \delta(\vartheta - \vartheta_s) \quad \text{Scene} \]

\[ \tilde{x}(\vartheta) = \sum_{s=1}^{S} x_s \mathcal{W}_s(\vartheta - \vartheta_s) \]

\( \mathcal{W}_s \) is a window centered at 0 e.g.

\[ w_s(\vartheta) = \exp \left\{ -\frac{\vartheta^2}{\sigma_s^2} \right\} \]

where \( \sigma_s^2 \) corresponds to the CRB
**CHALLENGE II: CONTINUOUS RECOVERY**

A new performance measure (to be discussed!)

Let $\mathbf{x}$ be the $S$-dimensional vector composed of the $x_s$.
For each $s = 1 \ldots S$ find a 'partner' $\tilde{x}_s$ and arrange them to a vector $\tilde{\mathbf{x}}$. The remaining error is defined as $\epsilon = \|\mathbf{x} - \tilde{\mathbf{x}}\|_2$.
The partner to $x_s$ is found as follows:

$$\rho(s) = \arg\max_{s'} \{\Re\{x_s w_s (\vartheta_s - \hat{\vartheta}_{s'}) \tilde{x}_{s'}^*\}\}$$

$$\tilde{x}_s := \tilde{x}_{\rho(s)} w_s (\vartheta_s - \hat{\vartheta}_{\rho(s)})$$

1. Scene point with window
   $$x_s w_s (\vartheta - \vartheta_s)$$

2. Estimated scene
   $$\hat{x}(\vartheta) = \sum_{s' = 1}^{S'} \hat{x}_s \delta(\vartheta - \hat{\vartheta}_{s'})$$

3. Real part of product of (1) and (2)*
CHALLENGE II: CONTINUOUS RECOVERY
Ways to overcome the grid bondage

1. Refinement of grid
2. Gradient based
3. Adaptive raster points


**CHALLENGE II: CONTINUOUS RECOVERY**

1. **Approach: Refinement of grid**

   Computed with ‘spgl1’

   Resolution cell / raster = 1.0

   ![Graph](image)

   - MF
   - OMP
   - BPDN

   Rayleigh resolution/raster size = 1

   N0 = 500  # resolution cells
   N = N0 ... 5000
   Oversampling = 1 ... 10
   M = 200
   S = 10
   # Simulations = 400
   dBnoise=-25;
CHALLENGE II: CONTINUOUS RECOVERY

1. Approach: Refinement of grid

Computed with ‘spgl1’

Resolution cell / raster = 4.0

N0 = 500    # resolution cells
N = N0 ... 5000
Oversampling = 1 ... 10
M = 200
S = 10
# Simulations = 400
dBnoise=-25;
CHALLENGE II: CONTINUOUS RECOVERY

1. Approach: Refinement of grid

Computed with ‘spgl1’

Resolution cell / raster = 7.0

N0 = 500  # resolution cells
N = N0 ... 5000
Oversampling = 1 ... 10
M = 200
S = 10
# Simulations = 400
dBnoise=-25;
CHALLENGE II: CONTINUOUS RECOVERY

1. Approach: Refinement of grid

Computed with ‘spgl1’

Resolution cell / raster = 10.0

- N0 = 500    # resolution cells
- N = N0 ... 5000
- Oversampling = 1 ... 10
- M = 200
- S = 10
- # Simulations = 400
- dBnoise=-25;
CHALLENGE II: CONTINUOUS RECOVERY
Performance measure for decreasing grid spacing

Computed with ‘spgl1’

- BPDN
- OMP
- MF

Remaining error (dB)

Rayleigh resolution/raster size

Performance grows with decreasing grid spacing.

OMP performs better than BPDN for decreasing grid spacing.
CHALLENGE II: CONTINUOUS RECOVERY

Performance measure for decreasing grid spacing

Don’t care about coherency (and RIP, null space property,...) !?
CHALLENGE II: CONTINUOUS RECOVERY

2. Approach: Add Taylor components

\[ y = \sum_{s=1}^{S} w_s \phi_s(\vartheta_s) + n \]

\[ = \sum_{s=1}^{S} w_s \phi_s(\vartheta_s + \Delta \vartheta_2) + n \]

\[ \approx \sum_{s=1}^{S} w_s \left[ \phi_s(\vartheta_s) + \Delta \vartheta s \phi_s(\vartheta_s) + \frac{1}{2} \Delta \vartheta^2 s \phi_s(\vartheta_s) + \ldots \right] + n \]

Linear approximation

Addition of new columns (gradients) to the sensing matrix

with \( z_s = w_s \Delta \vartheta_s \)
CHALLENGE II: CONTINUOUS RECOVERY

2. Approach: Add Taylor components

\[ y = Bw + Cz + n \]
\[ = Ax + n \]
with \( A = (B \quad C) \), \( x = \begin{pmatrix} w \\ z \end{pmatrix} \)

\[ \Delta \hat{\mathbf{v}}_s = \mathcal{R} \left\{ \frac{\hat{Z}_s}{\hat{\mathbf{w}}_s} \right\} \]
\[ \hat{\mathbf{v}}_s = \overline{\mathbf{v}}_s + \Delta \hat{\mathbf{v}}_s \]

A case for block-sparse recovery!

C. Ekanadham, D. Tranchina and E. P. Simoncelli, "Recovery of Sparse Translation-Invariant Signals With Continuous Basis Pursuit, 2011"
2. Approach: Add Taylor components

Reconstruction with mixed $\ell_1$-$\ell_2$-minimization

Coherence of the combined sensing matrix $\approx 0.6$
CHALLENGE II: CONTINUOUS RECOVERY

2. Approach: Add Taylor components, performance

N = 500
S = 5
Noise = -25 dB
Iteration:
- The grid points are shifted according to the actual estimation of displacements
- Re-applied sparse recovery
- Next iteration

Especially interesting for target tracking

Application example: Passive radar network (PCL), Block-sparse recovery
3. Approach: Adaptive grid

- Fixed search grid for the detection of new airplanes
- Dynamic track grid for tracked airplanes, basis for evaluating the remainders by projection
- Fine estimate of positions (here obtained by a second order Taylor approximation) can be integrated into the BOMP iteration

CHALLENGE II: CONTINUOUS RECOVERY

3. Approach: Adaptive grid, tracking

True positions of airplanes marked by blue circles

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
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</thead>
<tbody>
<tr>
<td>$f_0$</td>
<td>400</td>
<td>MHz</td>
</tr>
<tr>
<td>$\lambda_0$</td>
<td>0.75</td>
<td>m</td>
</tr>
<tr>
<td>Bandwidth</td>
<td>0.07</td>
<td>MHz</td>
</tr>
<tr>
<td>Resolution</td>
<td>2.142</td>
<td>m</td>
</tr>
<tr>
<td>$N_{Tx}$</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>$N_{Rx}$</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>$N_{array}$</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$L$ (number sensors)</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>$N_k$</td>
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<td></td>
</tr>
<tr>
<td>$M_{total}$</td>
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<td></td>
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<tr>
<td>Image</td>
<td>151 x 114</td>
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<tr>
<td>$N$</td>
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</tr>
<tr>
<td>SNR</td>
<td>20 dB</td>
<td></td>
</tr>
<tr>
<td>Simmax</td>
<td>500</td>
<td></td>
</tr>
</tbody>
</table>

Computed with BOMP
Challenge III: RANDOM (?) PROJECTION
CHALLENGE III: RANDOM (?) PROJECTION
Famous example: Rice Univ single-pixel camera

DMD: Digital micromirror device
RNG: Random number generator
CHALLENGE III: RANDOM (?) PROJECTION

Output

Measurements

Inputs

Signals at sensor (e.g. at aperture)

Scene (e.g. reflectivity)

Features

Projection

Signal model

Representation basis

Features

Coefficients

Selection

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CHALLENGE III: RANDOM PROJECTION (?)
Applied to array antennas

Sparing elements, channels and costs

SNR-loss!

(Random) projection
THEOREMS FOR SPARSE RECONSTRUCTION
Random selection of sensing waveforms

THEOREM of Candés and Romberg

Let an $S$-sparse scene $\rho$ with the coefficient vector $x$ be given with respect to an orthonormal representation basis $B$. With uniform probability let $M$ sensor waveforms be drawn from the orthonormal sensing basis $S$, forming the thinned sensing matrix $\tilde{S}$ and $A = \tilde{S}B$. Then the solution of the basis pursuit with $y = Ax$ is exact with probability $P \geq 1 - \epsilon$, if the following condition holds:

$$M \geq C\mu^2(S, B)S \ln\left(\frac{N}{\epsilon}\right)$$

with an appropriate constant $C$. 
CHALLENGE III: RANDOM PROJECTION (?)

Why to use random projections / selections?

- Just to be compatible to the theorems?

Deterministic projections / selections?

1. Heuristic choice
2. Optimum sparse ruler
3. Random search

Some papers about deterministic dimension reduction


CHALLENGE III: RANDOM PROJECTION (??)
Naive guess of a deterministic thinning

Example
Positions =
[0  1  3  6
 10 15 21 28
36 45 55 66
78 91 105 120]

M = 16
N = 121

Computed with ‘spgl1’
CHALLENGE III: RANDOM PROJECTION (?)
Naive guess of a deterministic thinning

Example
Positions =

\[
\begin{bmatrix}
0 & 1 & 2 & 3 & 4 & 6 & 8 \\
10 & 12 & 15 & 18 & 21 & 24 \\
28 & 32 & 36 & 40 & 45 & 50 \\
55 & 60 & 66 & 72 & 78 & 84 \\
91 & 98 & 105 & 112 & 120 \\
128 & 136 & 144 & 153 & 162 \\
171 & 180 & 190 & 200 & 210 \\
220 & 231 & 242 & 253 & 264
\end{bmatrix}
\]

M = 45

Computed with ‘spgl1’
CHALLENGE III: RANDOM PROJECTION (?)
Optimum sparse ruler

Example
Positions = [0, 1, 2, 5, 10, 15, 26, 37, 48, 54, 60, 66, 67, 68]
M = 14
N = 69

Computed with ‘spgl1’
CHALLENGE III: RANDOM PROJECTION (?)
Random search for optimum selection

- 200 random selections
- For each selection
  - Simulation of 200 scenes and reconstructions
- Selection with maximum probability of success fixed
- Further 400 simulations of scenes and reconstructions
- Comparison with non-optimized random selection

![Graph showing relative frequency of success over iterations](image.png)
CHALLENGE III: RANDOM PROJECTION (?)
Optimized random selection

Example
M = 100
N = 500
optimized for S=14
Challenge IV: LARGE SCENES / RAWDATA

For radar techniques as SAR the sensing matrices are for common scene sizes much too large to apply CS algorithms.

We propose a mosaicing technique based on ‘pre-focus’
CHALLENGE IV: LARGE SCENES / RAWDATA

Principle of pre-focusing / mosaicing

See also:

The aim is now to choose $P$ in such a way that $\tilde{A}$ obtains the form of a band matrix.
CHALLENGE IV: LARGE SCENES / RAWDATA

Principle of pre-focusing / mosaicing

- Contributing scene indices
- Used measurements for one mosaic piece
- Pre-focused sensing matrix
- This part of the reconstructed coefficient vector is used for mosaicing
To achieve approximately the form of a band-matrix, the data in the (spatial) frequency domain are pre-focused by a low-pass.

This has to have very low sidelobes, and an adequate passband interval large enough to preserve enough information.
Simulation: Comparison of a direct and a mosaiced recovery

N = 4000, M = 2000
S = 20, #Segments = 5
CHALLENGE IV: LARGE SCENES / RAWDATA

Principle of pre-focusing / mosaicing (2D)

- Real data recorded by Fraunhofer AER-II
- Size of the processed SAR-Image: 1991 x 751 = 1 495 241 Pixel
- Number of Mosaic-Pieces: 35 x 35 = 1225

![Diagram](image-url)
Overview image

Conventional processed

CS image
MOSAICING FOR CS-PROCESSING OF A SAR-IMAGE

Fine image grid + CS = super resolution!
Challenge V: NON-LINEAR SENSING
We regard a material probe composed of $K$ homogeneous lossless plane plates with different permittivities with relative dielectric constants $\varepsilon_1, \ldots, \varepsilon_K$ which are assumed to be constant over the measured frequency range and thicknesses $d_1, \ldots, d_K$ which are unknown.

There are only a few layers.

The $S$-parameters are measured over a range of frequencies.

Determine $\varepsilon_1, \ldots, \varepsilon_K$ and $d_1, \ldots, d_K$!
CHALLENGE V: NON-LINEAR SENSING
Example: $\varepsilon$-layer retrieval (simulation)

Original $\varepsilon$-layers and local reflection coefficients

Measured S-parameters in time domain
CHALLENGE V: NON-LINEAR SENSING
Example: $\varepsilon$-layer retrieval

**CS-solution:** The model of the probe is divided into $N$ thin slices of equal relative electrical lengths $\Delta L$, within which a constant $\varepsilon$ is assumed.

**Chain matrix for $\varepsilon$-jump:**

$$C_{\text{jump}}(\rho) = \frac{1}{\sqrt{1 - \rho^2}} \begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}$$

$$\rho = \frac{1}{\sqrt{\varepsilon_n + 1}} - \frac{1}{\sqrt{\varepsilon_n}}$$

**Chain matrix for passing the slice:**

$$C_{sl}(f) = \begin{pmatrix} q(f) & 0 \\ 0 & q^*(f) \end{pmatrix}$$

$$q(f) = \exp \left\{ -j2\pi \frac{f}{f_0} \Delta L \right\}$$
CHALLENGE V: NON-LINEAR SENSING

Example: $\varepsilon$-layer retrieval

Chaining the chain-matrices:

\[
C(f)(\rho) = C_{\text{jump}}(\rho_{N+1})C_{\text{sl}}(f) \ldots C_{\text{sl}}(f)C_{\text{jump}}(\rho_2)C_{\text{sl}}(f)C_{\text{jump}}(\rho_1)
\]

\[
S_{11}(f, \rho) = -\frac{C_{21}(f)}{C_{22}(f)}, \quad S_{21}(f, \rho) = \frac{1}{C_{22}(f)}.
\]

Idea of CS-recovery of the internal reflection coefficients:

Vector of measurements over the frequencies:

\[
z = \varphi(\rho) + n
\]

\[
\approx \varphi(\hat{\rho}) + \nabla\varphi(\hat{\rho})(\rho - \hat{\rho}) + n
\]

\[
y(\hat{\rho}) := z - \varphi(\hat{\rho}) + \nabla\varphi(\hat{\rho})\hat{\rho}
\]

\[
\approx \nabla\varphi(\hat{\rho})\rho + n
\]

\[
= A(\hat{\rho})\rho + n
\]
CHALLENGE V: NON-LINEAR SENSING

Example: $\varepsilon$-layer retrieval

Iteration: $\hat{\rho}_0 = \mathbf{0}$  $it = 0$

- calculate $\mathbf{A}(\hat{\rho}_{it})$
- solve $\mathbf{y}(\hat{\rho}) = \mathbf{A}(\hat{\rho}_{it})\hat{\rho}_{it+1} + \mathbf{n}$ via CS
- $it = it + 1$
- until a stop criterium is met.
CHALLENGE V: NON-LINEAR SENSING

Example: $\varepsilon$-layer retrieval

Internal reflection coefficients

Reconstruction of $\varepsilon$-layers
Challenge VI: HIGHER LEVEL INFORMATION RETRIEVAL
CHALLENGE VI: HIGHER LEVEL INFORMATION RETRIEVAL
View behind the curtain – Raw data to information converter
CHALLENGE VI: HIGHER LEVEL INFORMATION RETRIEVAL

Extraction of scatterers at straight lines in ISAR data

- The scatterers on man made targets are often arranged along straight lines

- Traditional approach:
  - Form an image
  - Apply methods like Hough-transform to identify straight lines

- Our approach:
  - Find a small number of lines explaining the measurements due to scatters placed on these lines largely!
  - A case for block sparse recovery
A finite set of potential lines has to be provided as well as a set of points along each line.

Model for the measurements, arranged as a column vector:

$$z = \sum_{\nu=1}^{N_l} \sum_{\mu=1}^{M(\nu)} s_{\nu\mu} x_{\nu\mu} + n$$

- Number of lines
- Number of grid points on lane $\nu$
- Coefficients (reflectivity)
- Signal vector for point $\mu$ on line $\nu$
- Noise
CHALLENGE VI: HIGHER LEVEL INFORMATION RETRIEVAL
Extraction of scatterers at straight lines in ISAR data

Model for the measurements, arranged as a column vector:

\[
\hat{x} = \text{argmin} \sum_{\nu=1}^{N_l} \sqrt{\sum_{\mu=1}^{M(\nu)} |x_{\nu\mu}|^2}
\]

subj. to \[\|z - \sum_{\nu=1}^{N_l} \sum_{\mu=1}^{M(\nu)} s_{\nu\mu} x_{\nu\mu}\|_2 \leq \sigma.\]

- I1 norm of rms vector
- Amplitude rms of line n (I2 norm)
- Rest not explained by sparse model
Goal (sparsity in the occupied planes): Find as few as possible lines explaining the measurement with a remaining error at noise level!
*Grid of potential lines and points required!*

Mixed norm approach:
Minimize

\[
\frac{\ell_1}{\ell_2} - \sum_{\nu=1}^{N_p} \sqrt{\sum_{\mu=1}^{M(\nu)} |x_{\nu\mu}|^2} \quad \text{subj. to} \quad \|z - \sum_{\nu=1}^{N_p} \sum_{\mu=1}^{M(\nu)} s_{\nu\mu} x_{\nu\mu}\|^2 \leq \sigma^2.
\]

- \(l_1\) norm of rms vector
- Amplitude rms of line \(n\) (\(l_2\) norm)
- Rest not explained by sparse model
Alternative: Block Orthogonal Matching Pursuit (BOMP)

- \( \text{it} = 1 \)
- Find line with maximum accumulated energy
- Iterate
  - Calculate remainder for the measurement projected to the space spanned by the signals for the points of all planes found until now.
  - Find plane with maximum energy with regard to the remainder
  - \( \text{it}=\text{it}+1 \)
- Until the rest can be explained by noise
CHALLENGE VI: HIGHER LEVEL INFORMATION RETRIEVAL
Extraction of scatterers at straight lines in ISAR data

Lines and points on lines

Principle of line-grid and point-grids on the lines
CHALLENGE VI: HIGHER LEVEL INFORMATION RETRIEVAL
Extraction of scatterers at straight lines in ISAR data

Simulation

Original scene  Fourier reconstruction  CS-lines reconstr.
CHALLENGE VI: HIGHER LEVEL INFORMATION RETRIEVAL
Extraction of scatterers at straight lines in ISAR data

ISAR-Image of a satellite, recorded by the FHR-radar TIRA
CHALLENGE VI: HIGHER LEVEL INFORMATION RETRIEVAL
Extraction of scatterers at straight lines in ISAR data
Thank you for listening!