Blind Deconvolution and Compressed Sensing

Dominik Stöger

Technische Universität München,
Department of Mathematics,
Applied Numerical Analysis

dominik.stoeger@ma.tum.de

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Overview

Introduction and Problem Formulation

Recovery guarantees

Proof sketch

Discussion
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Discussion
Blind Deconvolution

- bilinear inverse problem: $z = B(x, y)$
- ambiguities, constraining $x$ and/or $y$

Many applications:
- imaging (blind deblurring)
- radar, e.g., ground penetrating radar (GPR), radar imaging
- speech recognition
- wireless communication
Blind Deconvolution and Demixing

A problem in Wireless Communication:

- $r$ different devices
- device $i$ delivers message $m_i$
- **Linear encoding:**
  $$x_i = C_i m_i \text{ with } C_i \in \mathbb{R}^{L \times N}$$
- **Channel model:**
  $$w_i = B_i h_i, \text{ where } B_i \in \mathbb{R}^{L \times K}$$
- **Received signal:**
  $$y = \sum_{i=1}^{r} w_i \ast x_i \in \mathbb{R}^L$$

**Goal:** recover all $m_i$ from $y$
Assumptions on $B_i$ and $C_i$

\[ y = \sum_{i=1}^{r} w_i \ast x_i = \sum_{i=1}^{r} B_i h_i \ast C_i m_i \]

- Assume $w_i$ is concentrated on the first few entries, i.e., $B_i$ extends $h_i$ by zeros
- (Our analysis will include more general $B_i$)
- Choice of $C_i$ is arbitrary $\Rightarrow$ randomize
- Choose $C_i$ to have i.i.d. standard normal entries
Lifting

- There are unique linear maps $A_i : \mathbb{R}^{K \times N} \to \mathbb{R}^L$ such that for arbitrary $h_i$ and $m_i$

  $w_i \ast x_i = B_i h_i \ast C_i m_i = A_i (h_i m_i^*) = A_i (Y_i)$

- Low rank matrix recovery problem

  $y = \sum_{i=1}^{r} A_i (h_i m_i^*) = A (X_0)$,

  where

  $X_0 = (h_1 m_1^*, \cdots, h_r m_r^*)$
Overview

Introduction and Problem Formulation

Recovery guarantees

Proof sketch

Discussion
A convex approach for recovery

- $r = 1$ investigated in [Ahmed, Recht, Romberg, 2012]
- $r \geq 1$ semidefinite program (SDP) [Ling, Strohmer, 2015]

$$\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{r} \|Y_i\|_* \\
\text{subject to} & \quad \sum_{i=1}^{r} A_i(Y_i) = y. \\
\end{align*} \quad \text{(SDP)}$$

- $\| \cdot \|_*$: nuclear norm, i.e., the sum of the singular values
- Recovery is guaranteed with high probability, if

$$L \geq Cr^2 \left( K + \mu_h^2N \right) \log^3 L \log r$$

- coherence parameter: $1 \leq \mu_h = \max_i \frac{\|\hat{h}_i\|_\infty}{\|h_i\|_2} \leq \sqrt{K}$
Recovery guarantees

Linear scaling in $r$

- Number of degrees of freedom: $r(K + N)$
- Recovery guarantee of Ling and Strohmer (up to log-factors)
  \[ L \geq Cr^2 (K + \mu^2 h N) \log \cdots \]
- Optimal in $K$ and $R$, suboptimal in $r$
- Conjecture by Strohmer: Number of required measurements scales linear in $r$
- This is supported by numerical experiments
Main result

Theorem (Jung, Krahmer, S., 2016)

Let $\alpha \geq 1$. Assume that

$$L \geq C_\alpha r \left( K \log^2 K + N \mu_h^2 \right) \log^2 L \log (\gamma_0 r),$$

where

$$\gamma_0 = \sqrt{N \left( \log \left( \frac{NL}{2} \right) \right) + \alpha \log L}$$

and $C_\alpha$ is a universal constant only depending on $\alpha$. Then with probability $1 - \mathcal{O}(L^{-\alpha})$ the recovery program is successful, i.e. there exists $X_0$ is the unique minimizer of (SDP).

- (Near) optimal dependence on $K$, $N$, and $r$
Overview

Introduction and Problem Formulation

Recovery guarantees

Proof sketch

Discussion
Proof overview

Two main steps in the proof:

- Establishing sufficient conditions for recovery
  ⇒ approximate dual certificate
- Constructing the dual certificate via Golfing Scheme
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Crucial new ingredient for both steps:

**Restricted isometry property on 2r-dimensional space**

\[ T = \left\{ (u_1 m_1^* + h_1 v_1^*, \cdots, u_r m_r^* + h_r v_r^*) : \\
  u_1, \cdots, u_r \in \mathbb{R}^K, v_1, \cdots, v_r \in \mathbb{R}^N \right\} \]

Intuition for \( T \):
Directions of change when slightly varying the \( m_i \)'s and \( h_i \)'s
Restricted isometry property 1

Definition
We say that $\mathcal{A}$ fulfills the restricted isometry property on $T$ for some $\delta > 0$, if for all $X = (X_1, \cdots, X_r) \in T$

$$(1 - \delta) \sum_{i=1}^{r} \| X_i \|_F^2 \leq \left\| \sum_{i=1}^{r} \mathcal{A}_i (X_i) \right\|_{\ell_2}^2 \leq (1 + \delta) \sum_{i=1}^{r} \| X_i \|_F^2.$$ 

[Ling, Strohmer, 2015]: Each operator $\mathcal{A}_i$ acts almost isometrically on

$$T_i = \left\{ um_i^* + h_i v^* : u \in \mathbb{R}^K, \ v \in \mathbb{R}^N \right\}.$$ 

and $\mathcal{A}_i, \mathcal{A}_j$ are incoherent

$\Rightarrow r^2$-bottleneck
■ **Observe:**
Restricted isometry property for some $\delta > 0$ is equivalent to

$$\delta \geq \sup_{X \in T} \left| \left\| \sum_{i=1}^{r} A_i(X_i) \right\|_{\ell_2}^2 - \sum_{i=1}^{r} \|X_i\|_{F}^2 \right|$$

$$= \sup_{X \in T} \left| \left\| \sum_{i=1}^{r} A_i(X_i) \right\|_{\ell_2}^2 - \mathbb{E} \left[ \left\| \sum_{i=1}^{r} A_i(X_i) \right\|_{\ell_2}^2 \right] \right|.$$ 

■ **Suprema of chaos processes:** The last term can be bounded using results from [Krahmer, Mendelson, Rauhut, 2014].
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Recovery guarantees

Proof sketch

Discussion
Open questions

- Generalization to more general random matrices
- Faster algorithms
- What if only a few number of devices are active? Does one obtain better recovery guarantees?
- Generalization to sparsity assumption on $h$ (instead of a subspace assumption)